MST121 Chapter C1

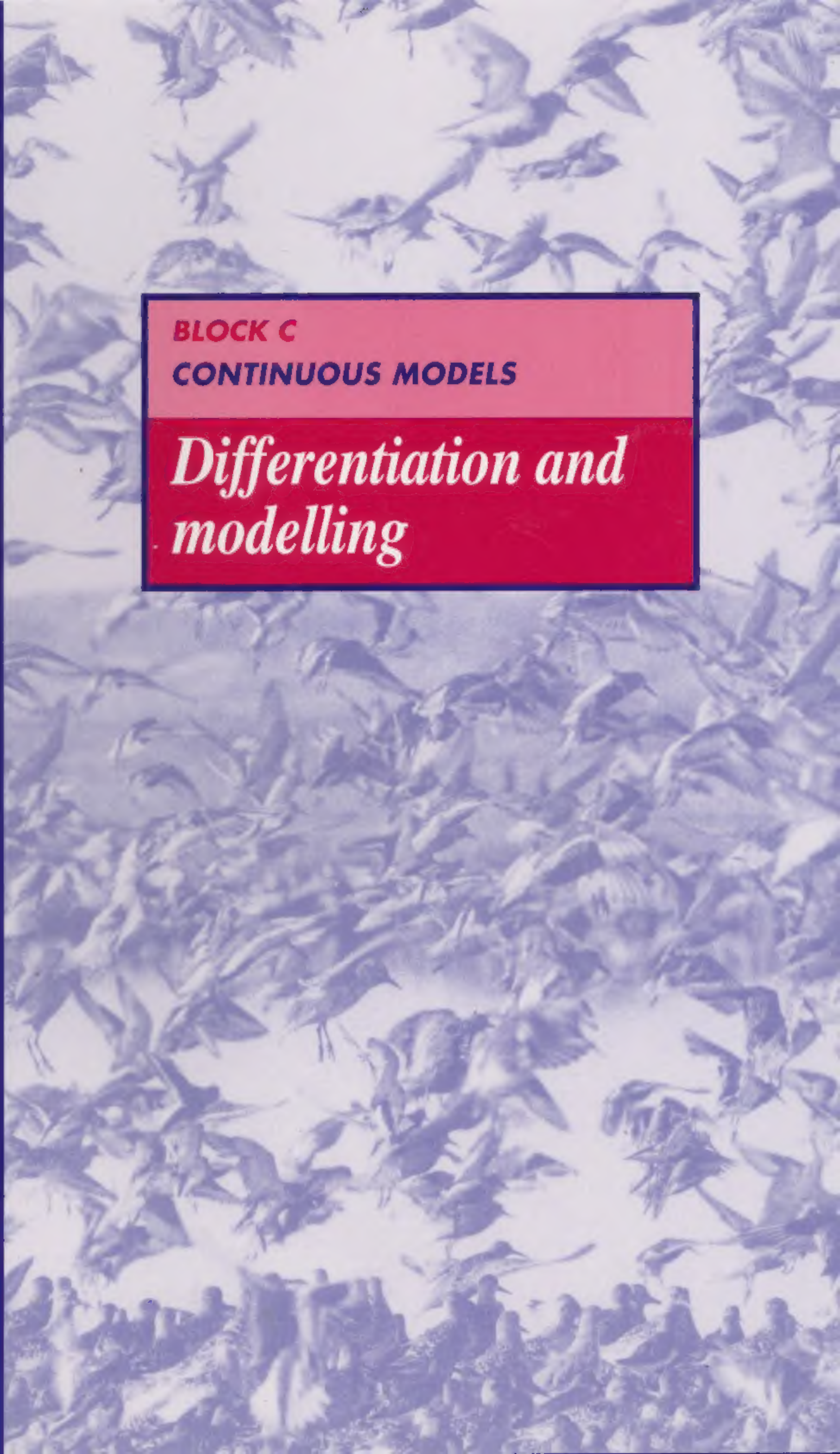
The Open University

A first level interdisciplinary course

Using Mathematics

CHAPTER

C1



BLOCK C

CONTINUOUS MODELS

Differentiation and modelling

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A first level interdisciplinary course

Using Mathematics

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Differentiation and modelling

Prepared by the course team

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About this course

This course, MST121 *Using Mathematics*, and the courses MU120 *Open Mathematics* and MS221 *Exploring Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MST121 uses the software program Mathcad (MathSoft, Inc.) and other software to investigate mathematical and statistical concepts and as a tool in problem solving. This software is provided as part of the course.

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Contents

Introduction to Block C

This applies particularly to Chapters B1 and B2.

The Summary of Block A, at the end of Chapter A3, described the difference between *discrete* models and variables, on the one hand, and *continuous* models and variables on the other. In much of Block B you used ideas from discrete mathematics, such as recurrence relations and sequences, to model phenomena that involve change.

In Block C you will again be modelling change, but this time in cases where the change is continuous or can reasonably be modelled as being so. Block C addresses questions such as the following: How fast is something changing? Is it increasing or decreasing? Does it ever stop changing, permanently or temporarily? How does the change accumulate? How can the change be described by an equation?

Chapter C1 concerns the rate at which variables change, whether the change is an increase or decrease and whether the change stops. A mathematical process called *differentiation* is introduced to help in analysing these matters, together with some illustrations of its practical applications.

Chapter C2 takes a different view towards change. The process of *integration* is introduced, as the reverse process of differentiation. Integration permits an accumulation of continuous changes to be described, and one significant application is to finding areas beneath graphs.

Integration also features importantly in Chapter C3. Here it is needed in order to solve a new type of equation, called a *differential equation*, which arises frequently in mathematical models of phenomena which change.

The Latin word *calculus* means a stone. The link is in the use of stones for counting.

The branch of mathematics which includes the study of differentiation and integration is called **calculus**, so Block C provides an introduction to calculus. The history of this branch of mathematics goes back to the second half of the seventeenth century, when Sir Isaac Newton (1642–1727) in England and Gottfried Wilhelm Leibniz (1646–1716) in what is now Germany both developed independently the basic ideas of calculus. The word 'calculus' here means 'a systematic method for solving a certain type of problem', and what Newton and Leibniz did was not so much to discover the basic ideas as to bring them together in a systematic way and then show how their systematic methods could be used to solve problems.

Unfortunately, a bitter argument grew up over who was the first to 'discover' calculus. This split the mathematicians of the day (and for decades afterwards) into two camps and hindered the development of the subject. In fact, one legacy of the dispute is still with us: there are several notations in current use for expressing exactly the same thing in calculus. One set of notations can be traced back to Newton's work and another to that of Leibniz. Other notations arose from more recent attempts to 'tidy up' the subject, following on from the greater understanding of calculus and its associated ideas that has developed since the time of Newton and Leibniz. You need to be able to recognise, understand and occasionally use all of these notations, even though you will probably settle eventually for using mainly one of the possible alternatives.

Many branches of mathematics, science and technology have made use of calculus, which indicates its central importance and usefulness. The chapters of this block refer to a range of contexts in order to illustrate the applications of calculus.

Study guide

There are five sections in this chapter. They are intended to be studied consecutively in five study sessions. Each section requires two to three hours.

The pattern of study for each session might be as follows.

Study session 1: Section 1.
Study session 2: Section 2.
Study session 3: Section 3.
Study session 4: Section 4.
Study session 5: Section 5.

Section 2 is the longest and Section 3, which contains some material that can be omitted if you are short of time, is the shortest.

Subsection 1.2 requires the use of an audio cassette player, and Subsection 3.3 requires the use of a video cassette player. Section 5 requires the use of the computer and Computer Book C.

Introduction

The use of a function to represent the relationship between a dependent variable and a single independent variable was introduced in Chapter A3, Subsection 1.1.

This chapter introduces an important topic in calculus called *differentiation*. If a dependent variable y is related to a particular independent variable x, which can vary continuously, then differentiation enables us to find the rate at which y changes as x changes. Put another way, this rate is the gradient of the graph defined by the equation $y = f(x)$ that relates the two variables. The expression for the rate of change in terms of x is called the *derivative* of $f(x)$.

Suppose, for example, that the motion of a car along a straight road is described by expressing its position relative to a fixed starting point in terms of the elapsed time. Then differentiation gives the rate of change of the car's position with time, which indicates just how fast the car travels during its journey. This rate of change, known as the velocity of the car, may itself vary with time. Differentiation permits the 'velocity function' of the car to be obtained directly from its 'position function', where the independent variable is time in each case.

Polynomial functions were introduced in Chapter A3, Section 5.

Section 1 defines the process of differentiation, illustrating its application first to quadratic functions and then to *power functions*. In Section 2, differentiation is applied to polynomial functions, following which you will see how differentiation can be employed to find the maximum or minimum value attained by a function. The process of finding these values is called *optimisation*, and is of use both for modelling purposes and for permitting the graphs of functions to be sketched quickly. While all the examples considered here involve polynomial functions, the principles apply also to many other types of function.

In Section 3, differentiation is applied to a wider range of functions, namely the trigonometric functions sin and cos, the exponential function exp and the natural logarithm function ln. The results are incorporated into a table giving the derivatives of important basic functions, to which are added two simple rules for differentiating functions defined as sums or constant multiples of those in the table. The alternative *Leibniz notation* for derivatives is described.

Section 4 gives three further rules for differentiation, which apply to functions that are defined as products, quotients and *composites* of the more basic functions considered earlier. The results in Sections 3 and 4 permit the differentiation of a very wide range of functions.

In Section 5 you will see how optimisation, based on differentiation, can be applied with the help of the computer, in the context of modelling various real-life problems.

1 What is differentiation?

To study Subsection 1.2 you will need an audio cassette player and Audio Tape 3.

In this section you will see how the idea of the *gradient* of a straight line can be extended to apply to any smooth graph. After an informal introduction in Subsection 1.1, you will see in Subsection 1.2 how to find the gradient for any quadratic graph. Subsection 1.3 formalises the process of finding the gradient for a curve, which is known as *differentiation*, and shows how this process can be applied to *power functions*.

1.1 Gradients of curves

A linear function which relates the dependent variable y to the independent variable x has a graph with equation of the form

$$y = mx + c.$$

Linear functions were introduced in Chapter A3, Subsection 1.2. The equations of lines and the definition of the slope or gradient of a line were given in Chapter A2, Subsection 1.1.

Here m is the *gradient* or *slope* of the graph, which represents the ratio of any change in y to the corresponding change in x. In other words, we have

$$m = \frac{\text{change in } y}{\text{corresponding change in } x} \quad \left(= \frac{\text{rise}}{\text{run}}\right),$$

as illustrated in Figure 1.1, for $m > 0$.

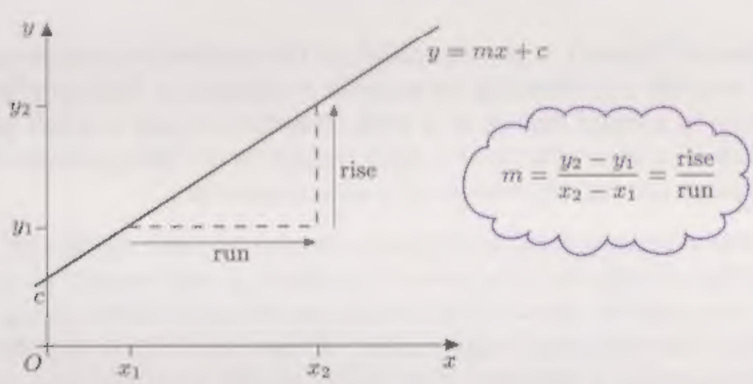


Figure 1.1 Gradient m of a linear function

If y decreases as x increases, then m is negative. If y does not change as x increases, then $m = 0$.

The graph of a linear function is equally steep at all points and so has constant gradient. This is not true for the graphs of non-linear functions. For example, the graph of the function $f(x) = x^2$ is 'flat' at the origin and becomes steeper as you move away from the origin to the left or right, as shown in Figure 1.2.

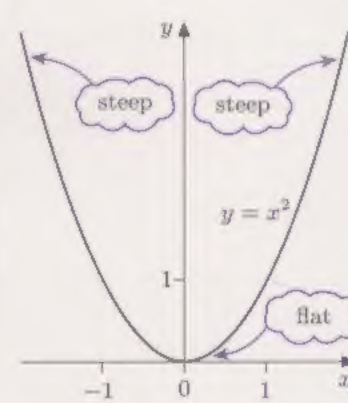


Figure 1.2 Graph of $y = x^2$

In general, we need to talk about the **gradient** or slope of a graph *at a point* on the graph. This is the gradient of the unique line that 'touches' the graph at the point, as shown in Figure 1.3(a) overleaf. This line, the one which most closely approximates the graph of $y = f(x)$ near the point, is called the **tangent** to the graph at the given point.

The word 'tangent' comes from the Latin verb *tangere*, meaning to touch.

One way to visualise the tangent at a point on the graph in Figure 1.3(a) is as follows. Think of progressively 'zooming in' on the portion of the graph which contains the particular point. This causes the portion of graph focused on to look straighter and straighter. Eventually it will look indistinguishable from a line segment, which is part of the tangent to the graph through the point.

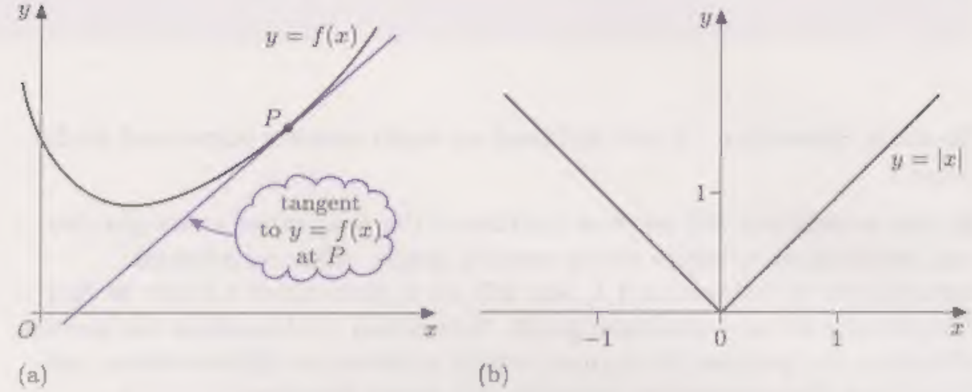


Figure 1.3 (a) Tangent to a graph at a point (b) No tangent to the graph at $(0, 0)$

You will see in this chapter that, for many functions f, the graph of f has a tangent at *every* point. Informally, a function is said to be *smooth* if it has this property. Any linear function is certainly smooth, since the tangent at each point coincides with the original straight-line graph. On the other hand, the function $f(x) = |x|$ is *not* smooth. The graph of this function is shown in Figure 1.3(b). Zooming in on the point $(0, 0)$ will always give a V-shape for the graph rather than a line segment. Hence there is no tangent to the graph at this point.

The gradients of tangents can often be expressed by a formula related to the rule of the function f. *Differentiation* is the process of finding this formula for all the tangent gradients, starting from the rule of the original function.

For the line of Figure 1.1, you can think of the gradient as a measure of *how fast* variable y is changing as variable x changes: a *large* gradient corresponds to a *rapid* change of y with respect to x, and a *small* gradient corresponds to a *slow* change of y with respect to x. The gradient of the line represents the *rate of change of y with respect to x.*

Similar remarks apply for graphs which are curved, such as the one in Figure 1.3(a), though here the rate of change of y with respect to x, being equal to the gradient at each point, varies as the point moves along the curve. The 'rate of change' relationship between variables is important in applications, and it is differentiation which enables us to find this rate of change at any point. You will see shortly how this process is carried out.

Differentiation: informal description

Differentiation is a process which enables you to find

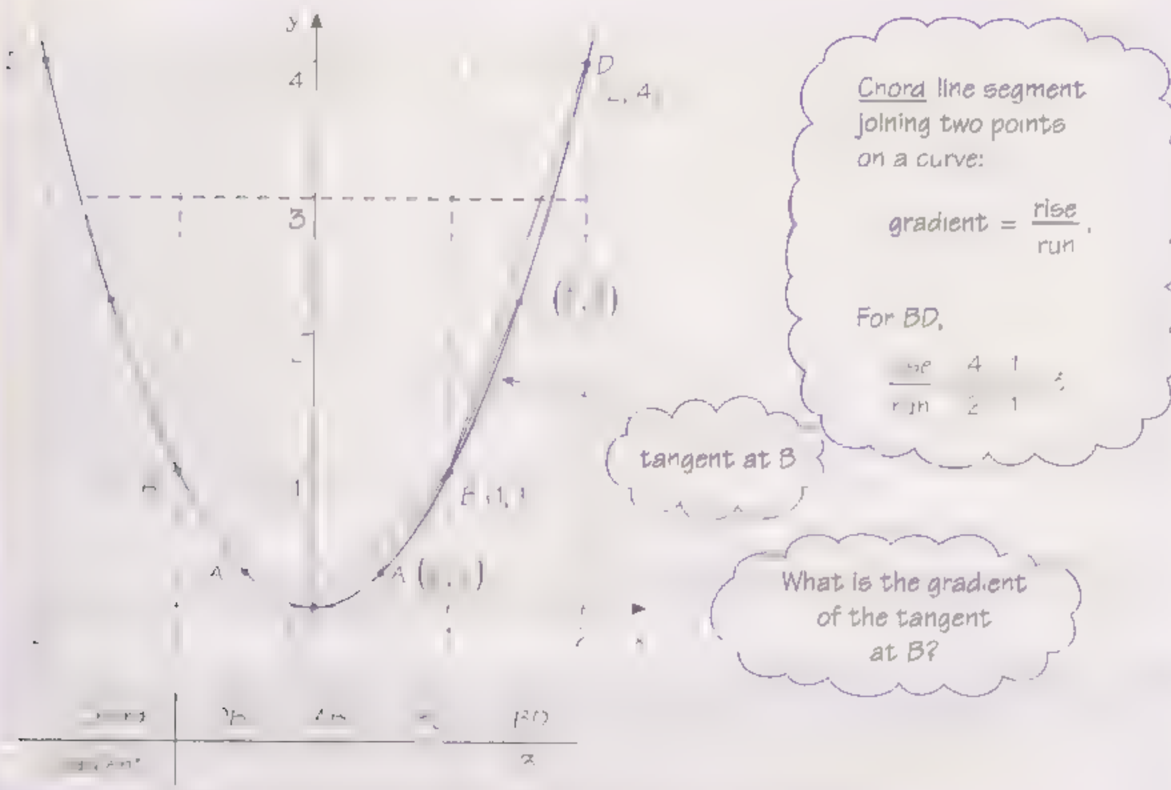
- the gradient of a graph;
- the rate at which one variable changes with respect to another.

1.2 Gradients of quadratic graphs

In the audio tape that follows, you will see that all quadratic functions are smooth and how to calculate the gradient at any point on any quadratic graph.

Now listen to Audio Tape 3, Band 1, 'Gradients of quadratic graphs'.

The graph of $y = x^2$



Gradient of tangent at (1, 1)

Gradient of chord with endpoints

$(1,1)$ and $(1 + h, (1 + h)^2)$ is

$$\frac{\text{rise}}{\text{run}} = \frac{(1+h)^2 - 1}{h}$$

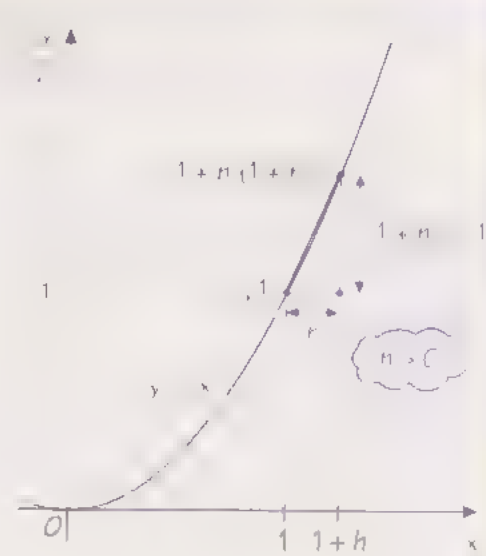
$$= \frac{[illegible]}{h}$$

$$= \frac{2h + h^2}{h} = 2 + h.$$

When h is small:

- ◊ gradient of chord is close to gradient of tangent at $(1,1)$;
- ◊ $2 + h$ is close to 2.

So gradient of tangent at $(1,1)$ is 2.

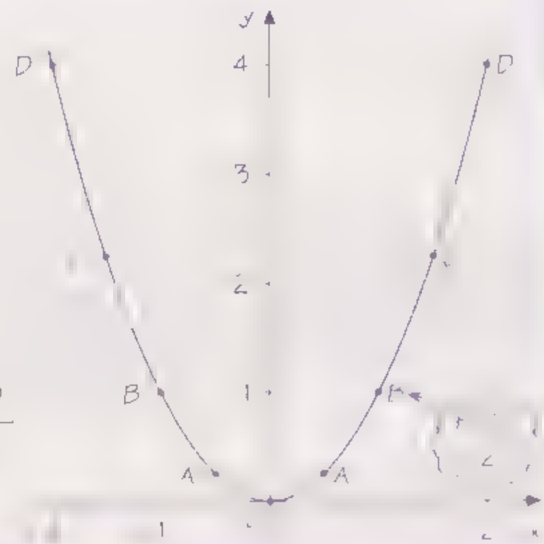


Gradient formula for $f(x) = x^2$

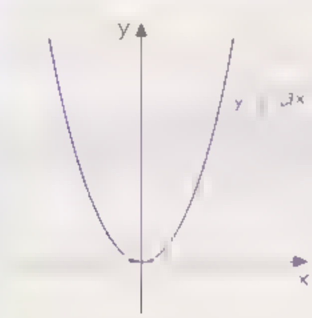
Gradient of $y = f(x)$ at $(x, f(x))$:
$f'(x) = 2x$.

gradient of tangent at $(x, f(x))$

Point	D	C	B	A	O	A	B	C	D
x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
$f'(x) = 2x$							2		



Gradient formula for $f(x) = ax^2 + bx + c$



gradient: $2ax$

gradient: b

gradient: 0

Gradient of $y = f(x)$ at $(x, f(x))$:
$f'(x) = 2ax + b$.

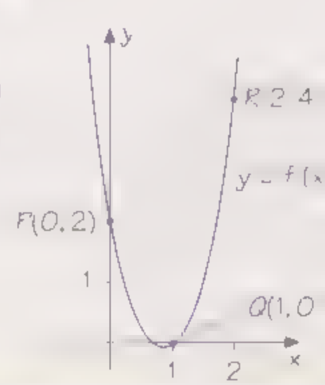
General quadratic gradient formula

Example

Let $f(x) = 3x^2 - 5x + 2$.

Find the gradient of $y = f(x)$ at P, Q and R.

$a = 3$, $b = -5$, $c = 2$



$f'(x) = 6x - 5$

Gradient at P: $6 \times 0 - 5 = -5$

Gradient at Q:

Gradient at R:

In Frame 4 of the audio tape it was stated that the gradient formula for a general quadratic function is given as:

$$\text{if } f(x) = ax^2 + bx + c, \text{ then } f'(x) = 2ax + b, \tag{1.1}$$

As mentioned on the audio tape, $f'(x)$ is pronounced as 'f dash of x' or 'f prime of x'.

where a, b and c are constants. This gives the value of the gradient of the graph of $y = f(x)$ at the point $(x, f(x))$, which is also the rate of change of y with respect to x at this point.

The following activities ask you to apply the quadratic gradient formula from result (1.1).

Activity 1.1 Evaluating the gradient formula

Use result (1.1) to write down an expression for $f'(x)$ in each of the following cases.

(a) $f(x) = x^2 + x + 7$

(b) $f(x) = 5x^2 - x$

(c) $f(x) = 5 + 3x - \frac{1}{2}x^2$

Solutions are given on page 54.

Activity 1.2 Finding gradients of quadratic graphs

Use result (1.1) to find the gradient at the point given on the graph of $y = f(x)$ in each of the following cases.

(a) $f(x) = x^2$, $(\frac{1}{2}, \frac{1}{4})$

(b) $f(x) = -x^2 + 1$, $(3, -8)$

(c) $f(x) = 3x^2 + 32x - 242$, $(1, -207)$

Solutions are given on page 54.

In Frame 2 of the audio tape you saw why the gradient of the function $f(x) = x^2$ at the point $B(1, 1)$ is 2, and it was asserted in Frame 3 that the gradient at a general point $(x, f(x))$ is given by

$$f'(x) = 2x. \tag{1.2}$$

We now justify this assertion, using reasoning similar to that in Frame 2. Figure 1.4 shows the graph of $y = x^2$, with a particular point (x, x^2) marked and a chord with one endpoint at (x, x^2).

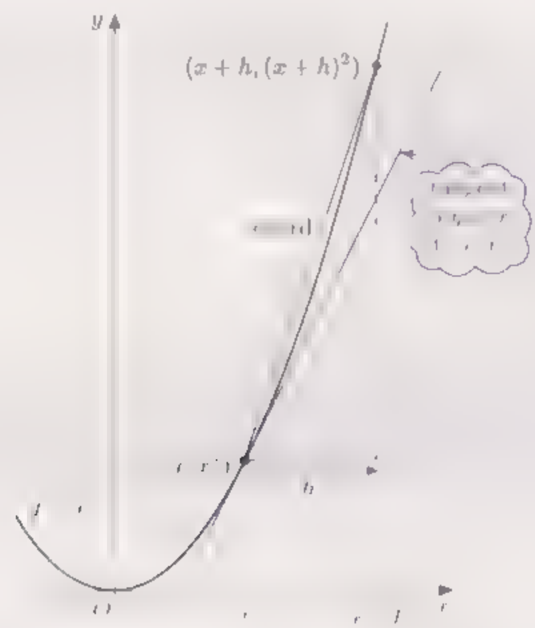


Figure 1.4 Finding the gradient for $y = x^2$ at a general point on its graph

In this argument, regard x as fixed while h varies. The quantity h can be positive or negative, but not zero.

The other endpoint of the chord is at $(x + h, (x + h)^2)$, a nearby point on the graph. For small values of h, the gradient of the chord is close to the gradient of the tangent at (x, x^2). The gradient of the chord is

$$\begin{aligned}\frac{\text{rise}}{\text{run}} &= \frac{(x+h)^2 - x^2}{(x+h) - x}\\ &= \frac{x^2 + 2xh + h^2 - x^2}{h}\\ &= \frac{2xh + h^2}{h}\\ &= 2x + h.\end{aligned}$$

Now the smaller h becomes, the closer the chord gradient is to the tangent gradient, and the closer $2x + h$ is to $2x$, so we conclude that the gradient of the tangent at (x, x^2) is $2x$, which confirms equation (1.2).

The argument to establish the gradient formula for the general quadratic function $f(x) = ax^2 + bx + c$, as given by result (1.1), is very similar to that just employed to verify formula (1.2) for the special case where $a = 1$, $b = 0$ and $c = 0$. Moreover, the approach adopted here, of first evaluating the gradient of a chord from $(x, f(x))$ to $(x + h, f(x + h))$ and then 'taking the limit' as h tends to zero, can be applied much more generally. This is taken up in the next subsection.

1.3 Differentiation of a function

A formula for the gradient of the graph of a general function can be found by applying the same approach as was used at the end of Subsection 1.2 for the particular case $f(x) = x^2$.

Suppose that f is any smooth function, so the tangent to its graph exists at all points. Figure 1.5 shows part of this graph, with a chord marked between the two points $(x, f(x))$ and $(x + h, f(x + h))$.

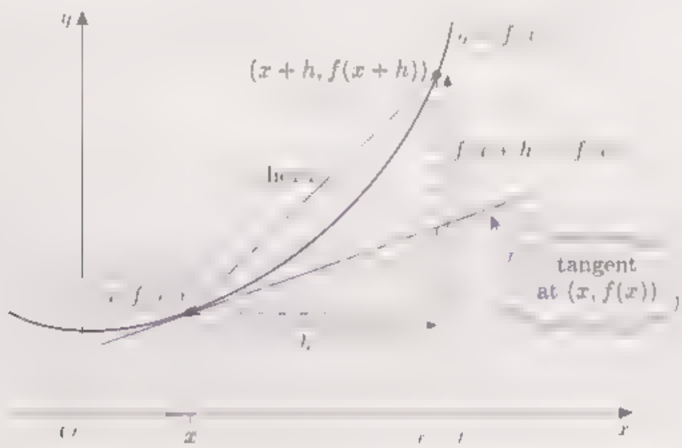


Figure 1.5 Finding the gradient for $y = f(x)$ at a general point on its graph

This chord has gradient

$$\frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{h}.$$

Once again, regard x as fixed while h varies. The quantity h can be positive or negative, but not zero.

Since f is a smooth function, its graph has a tangent at $(x, f(x))$, and the gradient of this tangent, denoted by $f'(x)$, must be the limit of the chord gradient values as h tends to zero. In other words, we have

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \qquad (1.3)$$

The symbol $\lim_{h \to 0}$ is read as 'the limit as h tends to 0'.

We call $f'(x)$, as defined by this formula, the *derivative* of f at the point x. For example, for the function $f(x) = x^2$, where as you have seen $f'(x) = 2x$, we have:

- the derivative of f at 1 is $f'(1) = 2$;
- the derivative of f at 0 is $f'(0) = 0$.

In general, the derivative $f'(x)$ depends both on the function f and on the point x. Hence the derivative defines a function, called the *derived function* of f and denoted by f'. For example, for the function $f(x) = x^2$, the derived function f' is defined by $f'(x) = 2x$.

Often the word 'derivative' is also used to refer to the derived function f'.

The process of finding the derived function or derivative of a given function f is called *differentiation*.

Differentiation, derivative and derived function

Let f be a function.

- The **derivative** $f'(x)$ at a point x in the domain of f is the gradient of the graph of f at $(x, f(x))$, given by

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right), \tag{1.3}$$

  provided that this limit exists.
- If the derivative exists at each point of the domain of f, then we say that f is **smooth**
- The process of finding the derivative $f'(x)$ is called **differentiation**
- The function f' defined by the process of differentiation is called the **derived function** of f.
- If $y = f(x)$, then $f'(x)$ is the **rate of change** of y with respect to x.

The verb associated with differentiation is 'to differentiate'. The meaning of this verb in calculus is therefore not the same as its meaning in everyday English (where it means 'to distinguish between' or 'to make different').

A word on domains is in order. The main emphasis in this course is on finding the rule for the derived function, f', of f. However, f' must also have a domain and this consists of those points in the domain of f where the derivative exists. In particular, if f is a smooth function, then the domain of f' is the same as that of f.

Depending on the particular purpose which you have in differentiating, you can regard the derivative as giving the gradient of the graph of the original function or the rate of change of the dependent variable with respect to the independent variable. Both interpretations are useful and widely applied.

So far you have seen how to differentiate quadratic functions. There are many types of smooth functions. In Sections 3 and 4 you will see the derived functions for a range of basic functions, together with some rules for finding derived functions for a wide class of functions.

Other than in this section, we shall not ask you to spend time in *justifying* these various results formally, since this is of secondary importance compared to being able to apply them appropriately. Such justifications involve starting in each case from definition (1.3), and we now look briefly at what this involves in a few simple cases.

Example 1.1 Differentiation from the definition

Using this definition directly is also called 'differentiating from first principles'.

Use equation (1.3) to differentiate the function $f(x)$ [illegible]

Solution

By equation (1.3), we need to consider the quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h},$$

where h is a non-zero number.

Now we have

$$\begin{aligned}(x+h)^3 &= (x+h)(x+h)^2\\ &= (x+h)(x^2+2xh+h^2)\\ &= x^3+3x^2h+3xh^2+h^3,\end{aligned}$$

so the quotient becomes

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}\\ &= \frac{3x^2h+3xh^2+h^3}{h}\\ &= 3x^2+3xh+h^2.\end{aligned}$$

Since $h \neq 0$, the possibility of division by zero does not arise.

For small values of h, the expression $3x^2+3xh+h^2$ is close to $3x^2$ because both $3xh$ and h^2 are small. Thus in the limit as $h \to 0$, we obtain

$$f'(x) = 3x^2.$$

The result of Example 1.1 can be applied to give the gradient at any point on the graph of $y = x^3$. For example, the points $(1, 1)$ and $(-1, -1)$ both lie on $y = x^3$, and the gradients at these points are respectively

$$f'(1) = 3 \quad \text{and} \quad f'(-1) = 3.$$

The tangents at these points are shown in Figure 1.6, together with the tangent to $y = x^2$ at $(1, 1)$, which has gradient 2, for comparison.

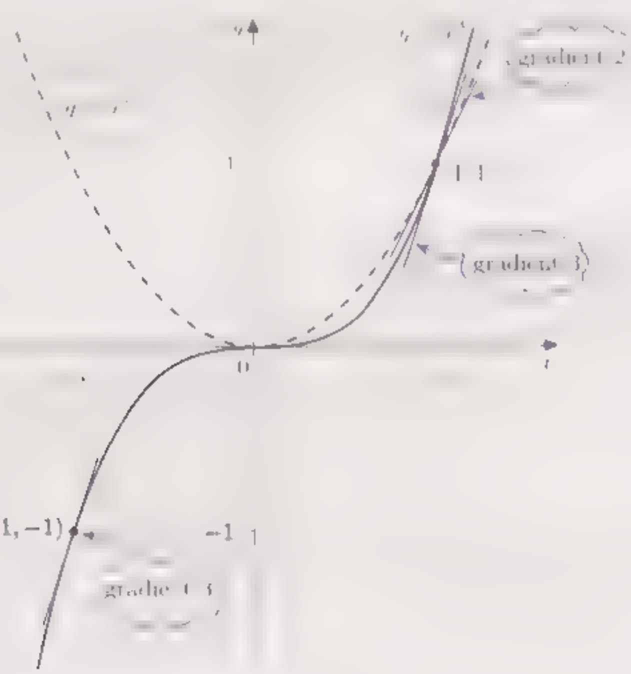


Figure 1.6 Gradients of tangents to the graphs of $y = x^3$ and $y = x^2$

Activity 1.3 Differentiation from the definition

(a) Use the expansion for $(x+h)^3$ from Example 1.1 to find the expansion for $(x+h)^4 = (x+h)(x+h)^3$.

(b) Use equation (1.3) and the result of part (a) to differentiate the function $f(x) = x^4$.

(c) What is the gradient of the graph of $y = x^4$ at the point $(\frac{1}{2}, \frac{1}{16})$?

Note that if $x = \frac{1}{2}$, then $y = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$. Hence the point $\left(\frac{1}{2}, \frac{1}{16}\right)$ lies on the graph.

Solutions are given on page 54.

You have now seen it established that:

- if $f(x) = x^2$, then $f'(x) = 2x$ (Subsection 1.2);
- if $f(x) = x^3$, then $f'(x) = 3x^2$ (Example 1.1);
- if $f(x) = x^4$, then $f'(x) = 4x^3$ (Activity 1.3).

In addition, you know that if $f(x) = x$ then $f'(x) = 1$, since the line $y = x$ has gradient 1, and that if $f(x) = 1$ then $f'(x) = 0$ because the line $y = 1$ is horizontal and hence has gradient zero. These results can be summarised in the single statement

Recall that $x^0 = 1$ and $x^1 = x$.

$$\text{if } f(x) = x^n, \text{ then } f'(x) = nx^{n-1} \quad \text{for } n = 0, 1, 2, 3, 4.$$

In fact, this statement holds not just for these particular values but for *any* real value of n, so we can write

$$\text{if } f(x) = x^n, \text{ then } f'(x) = nx^{n-1}, \tag{1.4}$$

where n is any real number. Functions of the form $f(x) = x^n$ are called **power functions**, so result (1.4) gives the rule for differentiating any power function. This result is often expressed as

The domain of a power function depends on the value of n. If n is a non-negative integer, then the domain is $\mathbb{R}$.

the derivative of x^n is nx^{n-1}.

Activity 1.4 Differentiation from a formula

Use result (1.4) to differentiate each of the following functions.

Recall that $1/x^3 = x^{-3}$ and $\sqrt{x} = x^{1/2}$.

(a) $f(x) = x^6$ (b) $f(x) = 1/x^3$ (c) $f(x) = \sqrt{x}$ $(x > 0)$

Solutions are given on page 54.

The concept of differentiation applies equally to situations where variables other than x and y are involved. In the course of mathematical modelling it will often be the case that other variable labels occur. For example, if s (in metres) is the position of a car moving along a straight road, measured from a fixed starting point, and t (in seconds) is the time since leaving the starting point, then the particular motion of the car may be described by $s = f(t)$ for some smooth function f. The derivative of this function gives the rate of change of the car's position with time, which is its velocity $v = f'(t)$, measured in metres per second ($\mathrm{m\,s^{-1}}$).

This derived function gives the *instantaneous* velocity of the car, which is what would be recorded at each time t on its speedometer.

In fact, this example permits a physical interpretation of the definition (1.3) for the derivative. If $s = f(t)$, then

$$\frac{f(t+h) - f(t)}{h}$$

is the *average* velocity (distance travelled divided by time passed) over the interval between time t and time $t+h$ (assuming $h > 0$). The instantaneous velocity $v = f'(t)$ at time t is the limit of such average velocities as the duration h of the time interval tends to zero.

Example 1.2 Finding the velocity

A car moving along a straight road has position $s = 30t - t^2$ metres, as measured in a given direction from a fixed point O, at time t seconds after it passes O.

(a) Find the corresponding velocity v (in $\mathrm{m\,s^{-1}}$) of the car at time t.

(b) What is the velocity of the car as it passes O?

(c) At what time does the car have velocity zero?

Solution

(a) In this case the position function is $s = f(t) = 30t - t^2$. This is a quadratic function (of t), so we may apply result (1.1) to obtain the velocity function

$$v = f'(t) = 30 - 2t.$$

(b) The car passes O when $t = 0$, at which time we have $v = f'(0) = 30$, so the velocity is then $30\ \mathrm{m\,s^{-1}}$.

(c) When $v = 0$, we have $30 - 2t = 0$, which has solution $t = 15$. Hence the car has velocity zero (is at rest momentarily) after 15 seconds.

For $t > 15$ the values of v are negative, so the car then moves back towards O. However, the model may be intended to apply only for $0 \le t \le 15$.

Activity 1.5 Finding the velocity

A stone dropped from rest has position $s = 5t^2$ metres below its starting point after t seconds.

(a) Find the velocity v (in $\mathrm{m\,s^{-1}}$) of the stone at time t.

(b) What is the velocity of the stone after 3 seconds?

(c) At what time does the stone have velocity $15\ \mathrm{m\,s^{-1}}$?

Solutions are given on page 54.

Summary of Section 1

This section has introduced:

- differentiation (informally) as a process which enables you to find either the gradient of a graph, or the rate at which one variable changes with respect to another;
- differentiation of a function f (formally) as the process of finding the derived function f' and derivative $f'(x)$, as defined by

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right); \qquad (1.3)$$

- the application of differentiation to quadratic functions and to power functions $f(x) = x^n$, where n is a real number.

Exercises for Section 1

Exercise 1.1

Use result (1.1) to find the gradient of the graph of $y = 6 + 5x - 4x^2$ at the point $(2, 0)$.

Exercise 1.2

(a) Show that if $x \neq 0$ and $x + h \neq 0$, then

$$\frac{1}{x+h} - \frac{1}{x} = -\frac{h}{(x+h)x}.$$

(b) Use equation (1.3) and the result of part (a) to differentiate the function

$$f(x) = \frac{1}{x}$$

(c) Verify that the result of part (b) can also be obtained by applying result (1.4).

Exercise 1.3

An aeroplane accelerating along a runway before take-off has travelled a distance $s = 3t^2 + t$ metres from the start of the runway after t seconds.

(a) Find the corresponding velocity v (in $\mathrm{m\,s^{-1}}$) of the aeroplane at time t.

(b) What is the velocity of the aeroplane after 10 seconds?

(c) At what time does the aeroplane have velocity $25\ \mathrm{m\,s^{-1}}$?

2 Graphs and optimisation

In Section 1 the process of differentiation was introduced. You saw how this process could be applied in certain cases to find formulas for the derivatives of functions, and so obtain values of the gradients at points on the corresponding graphs.

A major application of differentiation is in finding an *optimum* value of a function, that is, a highest or lowest value. In a modelling context this may represent the 'best' way of achieving some practical outcome, subject to certain constraints. For example, it might correspond to a maximisation of profit or a minimisation of resource expenditure.

Optimisation is discussed in Subsection 2.3. In preparation for this topic, Subsection 2.2 shows how differentiation can assist in understanding the structure of the graph of a given function, which is also useful in its own right. The examples here are based on simple polynomial functions. Subsection 2.1 shows how to find the derivatives of such functions.

Polynomial functions were introduced in Chapter A3, Section 5.

2.1 Differentiating polynomial functions

A polynomial function of degree n is a function whose rule has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0, \tag{2.1}$$

where $a_n \neq 0$. This includes the cases of linear functions ($n = 1$), quadratic functions ($n = 2$) and cubic functions ($n = 3$). For each of the first two of these cases, you saw in Section 1 how the corresponding derivative is obtained. Thus

$$f(x) = a_1x + a_0 \text{ has derivative } f'(x) = a_1$$

and

$$f(x) = a_2x^2 + a_1x + a_0 \text{ has derivative } f'(x) = 2a_2x + a_1.$$

The line $y = a_1x + a_0$ has gradient a_1. The derivative of a quadratic function was given, with different labels for the coefficients, in result (1.4).

For the general polynomial function given by equation (2.1), the derivative is

$$f'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1.$$

This result can be written concisely, using sigma notation, as

$$\text{if } f(x) = \sum_{i=0}^{n} a_ix^i, \text{ then } f'(x) = \sum_{i=1}^{n} ia_ix^{i-1}. \tag{2.2}$$

Sigma notation was introduced in Chapter B1, Subsection 1.2.

The lower limit in the summation for $f'(x)$ in result (2.2) can be written as $i = 1$ rather than $i = 0$, since the value of the term ia_ix^{i-1} for $i = 0$ is zero.

Note, from result (1.4) for the derivative of x^n, that result (2.2) is equivalent to the statement

$$\text{if } f(x) = \sum_{i=0}^{n} a_ix^i, \text{ then } f'(x) = \sum_{i=1}^{n} a_i(\text{derivative of } x^i), \tag{2.3}$$

which is an alternative way to remember the result.

Example 2.1 Differentiating a polynomial

Differentiate the polynomial function $f(x) = 2x^6 - 9x^4 + 3x$.

Solution

Comparing $f(x) = 2x^6 - 9x^4 + 3x$ with equation (2.1), we have $n = 6$, $a_1 = 3$, $a_4 = -9$, $a_6 = 2$ and $a_0 = a_2 = a_3 = a_5 = 0$.

From result (1.4), the derivative of x^6 is $6x^5$, the derivative of x^4 is $4x^3$ and the derivative of x $(= x^1)$ is 1 $(= x^0)$. From result (2.3), the derivative of the given polynomial function is

$$\begin{aligned} f'(x) &= 2(\text{derivative of } x^6) - 9(\text{derivative of } x^4) + 3(\text{derivative of } x) \\ &= 2(6x^5) - 9(4x^3) + 3(1) \\ &= 12x^5 - 36x^3 + 3. \end{aligned}$$

Activity 2.1 Differentiating polynomials

(a) Differentiate the polynomial function $f(x) = 6x^8 - 3x^5$.

(b) What is the gradient of the graph of $y = -x^3 + 5x^2 - 7x + 15$ at the point $(1, 12)$?

Solutions are given on page 54.

2.2 *Features of graphs*

In this subsection you will meet methods for identifying various key features of the graphs of functions. The examples involve polynomial functions, for which the derivative formula was given in the previous subsection. However, the degree of the polynomial function in each case will be no higher than 4.

A polynomial function of degree 4 is called a **quartic function**.

Several useful results will be stated. Note that these results apply to *any* smooth function, and not just to the types of polynomial function considered in the examples.

We look first at the graph of the cubic function

$$f(x) = 2x^3 + 3x^2 - 12x - 4,$$

which is shown in Figure 2.1.

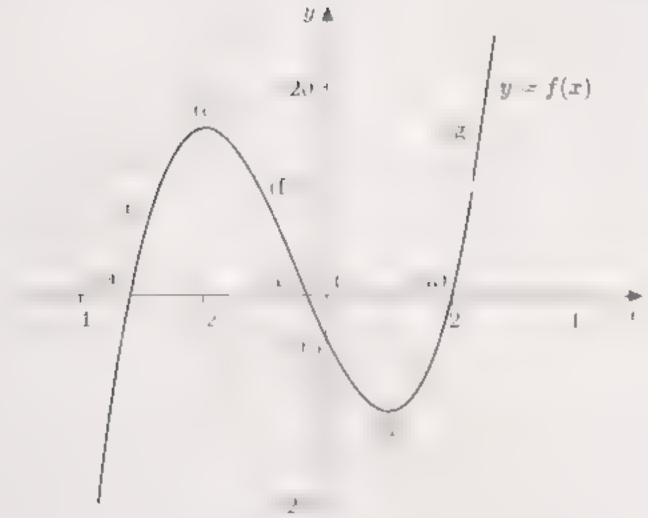


Figure 2.1 Graph of $y = 2x^3 + 3x^2 - 12x - 4$

Note that this and some other graphs in this chapter are drawn with unequal scales on the two axes

The labels in Figure 2.1 refer to the following important features of this graph.

(a) the graph crosses the x-axis at three points, approximately at $x = -3.2$, $x = -0.3$ and $x = 2.0$;

(b) the graph crosses the y-axis at one point, where $y = -4$;

(c) the graph has a 'peak' at approximately $(-2, 16)$;

(d) the graph has a 'trough' at approximately $(1, -11)$;

(e) to the left of the peak, the graph rises;

(f) between the peak and the trough, the graph falls;

(g) to the right of the trough, the graph rises.

In (e)–(g), and in similar statements throughout this subsection, it is assumed that the graph is traversed from left to right.

We now consider each of features (a)–(d), in turn.

The x-values where the graph crosses the x-axis are the solutions of the equation

$$f(x) = 0; \quad \text{that is,} \quad 2x^3 + 3x^2 - 12x - 4 = 0.$$

This is a cubic equation, and in general such equations cannot be solved straightforwardly by hand.

However, in this particular case, it turns out that $x = 2$ is a solution, from which the other two solutions can also be found if needed.

The y-value where the graph crosses the y-axis is $f(0) = -4$, which is the constant term in the polynomial.

A peak such as the one near $(-2, 16)$ in Figure 2.1 is called a *local maximum* of the function because it gives the greatest function value in the immediate vicinity of the point (the word 'local' is needed because the function may take greater values elsewhere). Similarly, the trough on the graph near $(1, -11)$ is called a *local minimum*.

The plural of 'maximum' is 'maxima', and the plural of 'minimum' is 'minima'.

In general, at local maxima and minima, the gradient is zero. In other words, the tangent at a local maximum or minimum is horizontal, as shown in Figure 2.2.

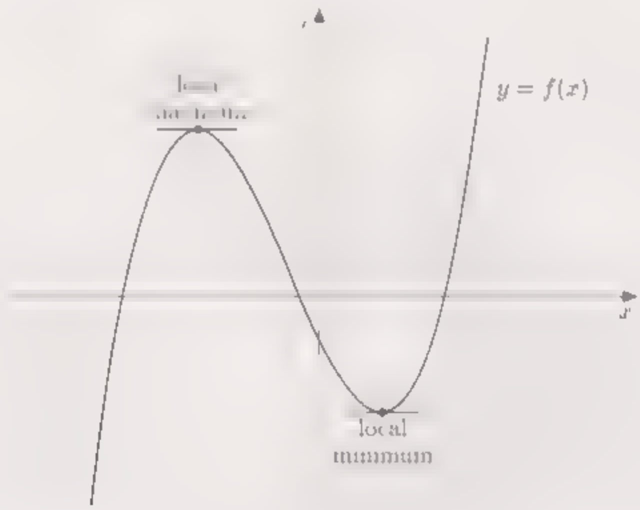


Figure 2.2 Horizontal tangents at local maximum and minimum

As you will see, not all stationary points are local maxima or minima.

Thus the local maxima and minima can be found by identifying those values of x for which $f'(x) = 0$, which are known as the *stationary points* of the function f.

Stationary point, local maximum and local minimum

Let f be a smooth function.

- The function f has a **stationary point** at $x = x_0$ if $f'(x_0) = 0$. The corresponding point $(x_0, f(x_0))$ on the graph of f is also called a stationary point.
- The function f has a **local maximum** at a stationary point $x = x_0$ if $f(x_0)$ is greater than any other function value in the immediate vicinity of x_0. The corresponding point $(x_0, f(x_0))$ on the graph of f is also called a local maximum.
- The function f has a **local minimum** at a stationary point $x = x_0$ if $f(x_0)$ is less than any other function value in the immediate vicinity of x_0. The corresponding point $(x_0, f(x_0))$ on the graph of f is also called a local minimum.

More precisely, there is a local maximum at $x = x_0$ if there is some open interval I, containing x_0, such that $f(x) < f(x_0)$ for any $x \neq x_0$ in I. A similar statement can be made for a local minimum, with the inequality replaced by $f(x) > f(x_0)$.

Example 2.2 Finding stationary points

(a) Differentiate the function

$$f(x) = 2x^3 + 3x^2 - 12x - 4,$$

whose graph was given in Figure 2.1.

(b) By solving the equation $f'(x) = 0$, find the stationary points of f.

Note that, in order to answer this type of question, it suffices to find the x-coordinate of each stationary point, unless you are explicitly asked to calculate also the corresponding function value.

Solution

(a) From results (1.4) and (2.3), the derivative of the given polynomial function is

$$\begin{aligned} f'(x) &= 2(3x^2) + 3(2x) - 12(1) \\ &= 6x^2 + 6x - 12 \\ &= 6(x^2 + x - 2). \end{aligned}$$

(b) The equation $f'(x) = 0$ is equivalent to

$$x^2 + x - 2 = 0; \quad \text{that is,} \quad (x+2)(x-1) = 0.$$

This equation has solutions $x = -2$ and $x = 1$. Thus the stationary points of f are at $x = -2$ and $x = 1$

Comment

By substituting the values $x = -2$ and $x = 1$ in turn into the rule for f, we find that $f(-2) = 16$ and $f(1) = -11$. In fact, the two stationary points correspond to the local maximum $(-2, 16)$ and the local minimum $(1, -11)$. This confirms the values given for features (c) and (d) following Figure 2.1.

Activity 2.2 Finding stationary points

(a) Differentiate the function

$$f(x) = x^3 - 6x^2 - 15x + 54.$$

(b) By solving the equation $f'(x) = 0$, find the stationary points of f.

Solutions are given on page 55.

Next we discuss the three features (e), (f) and (g) of the graph in Figure 2.1. These relate to the parts of the graph which rise or fall. To describe these features more formally, it is convenient to extend the concept of an increasing (or decreasing) function, as follows.

The concepts of 'increasing' and 'decreasing' were introduced in Chapter A3, Subsection 4.1, where they applied to the whole domain of a function.

Suppose that f is a real function and that the interval I is in the domain of f. Then f is said to be **increasing on** I if

for all x_1, x_2 in I, if $x_1 < x_2$, then $f(x_1) < f(x_2)$,

and f is said to be **decreasing on** I if

for all x_1, x_2 in I, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

For example, the function $f(x) = x^2$ is increasing on the interval $(0, \infty)$ and decreasing on the interval $(-\infty, 0)$; see Figure 2.3 overleaf.

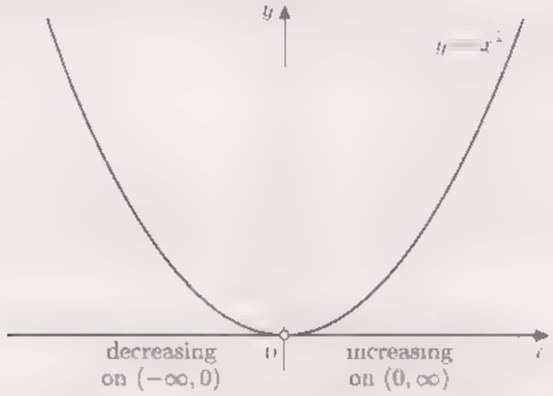


Figure 2.3 Intervals on which $f(x) = x^2$ is increasing/decreasing

We can often determine intervals on which a function f is increasing or decreasing by using differentiation. This is so because the derivative $f'(x)$ gives the gradient of the graph at the point $(x, f(x))$, and the *sign* of the gradient is related to whether the function is increasing or decreasing.

Note that in this criterion I is an *open* interval.

Increasing/Decreasing Criterion

Let I be an open interval in the domain of a smooth function f.

- ◇ If $f'(x) > 0$ for all x in I, then f is increasing on I.
- ◇ If $f'(x) < 0$ for all x in I, then f is decreasing on I.

In this display the common practice of writing 'for x in ...' to mean 'for all x in ...' has been used. This practice will be followed except in statements of definitions and results.

For example, consider the function $f(x) = x^2$ whose graph is shown in Figure 2.3. In this case

$$f'(x) = 2x, \quad \text{so} \quad \begin{cases} f'(x) > 0 & \text{for } x \text{ in } (0, \infty), \\ f'(x) < 0 & \text{for } x \text{ in } (-\infty, 0). \end{cases}$$

This corresponds to the increasing and decreasing properties of f shown in Figure 2.3, in accordance with the Increasing/Decreasing Criterion.

Example 2.3 Finding where a function is increasing/decreasing

In Example 2.2 you saw that the function $f(x) = 2x^3 + 3x^2 - 12x - 4$ has derivative

$$f'(x) = 6(x^2 + x - 2) = 6(x + 2)(x - 1).$$

(a) Show that

$$\begin{cases} f'(x) > 0 & \text{for } x \text{ in } (-\infty, -2), \\ f'(x) < 0 & \text{for } x \text{ in } (-2, 1), \\ f'(x) > 0 & \text{for } x \text{ in } (1, \infty). \end{cases}$$

(b) Use the Increasing/Decreasing Criterion to deduce from part (a) intervals on which f is increasing or decreasing.

Solution

(a) For $x < -2$, we have

$$x + 2 < 0 \quad \text{and} \quad x - 1 < -3 < 0,$$

so $f'(x) = 6(x + 2)(x - 1) > 0$ (since $f'(x)$ is the product of 6 and of two negative numbers).

For $-2 < x < 1$, we have

$$x + 2 > 0 \quad \text{and} \quad x - 1 < 0,$$

so $f'(x) < 0$ (since $f'(x)$ is the product of 6, one other positive number and one negative number).

For $x > 1$, we have

$$x + 2 > 3 > 0 \quad \text{and} \quad x - 1 > 0,$$

so $f'(x) = 6(x + 2)(x - 1) > 0$ (since $f'(x)$ is the product of 6 and of two other positive numbers).

(b) By the Increasing/Decreasing Criterion, f is increasing on $(-\infty, -2)$ and on $(1, \infty)$, but decreasing on $(-2, 1)$.

Activity 2.3 Finding where a function is increasing/decreasing

In Activity 2.2 you showed that the function $f(x) = x^3 - 6x^2 - 15x + 54$ has derivative

$$f'(x) = 3x^2 - 12x - 15 = 3(x + 1)(x - 5).$$

(a) Show that

$$\begin{cases} f'(x) > 0 & \text{for } x \text{ in } (-\infty, -1), \\ f'(x) < 0 & \text{for } x \text{ in } (-1, 5), \\ f'(x) > 0 & \text{for } x \text{ in } (5, \infty). \end{cases}$$

(b) Use the Increasing/Decreasing Criterion to deduce from part (a) intervals on which f is increasing or decreasing.

Solutions are given on page 55.

The connection between derivatives and intervals on which a function is increasing or decreasing can be used to provide a useful test for classifying a stationary point as a local maximum or a local minimum. This is stated in the box below and illustrated in Figures 2.4 and 2.5.

First Derivative Test

Suppose that x_0 is a stationary point of a smooth function f; that is, $f'(x_0) = 0$.

- If $f'(x)$ changes sign from positive to negative as x increases through x_0, then f has a local maximum at x_0.
- If $f'(x)$ changes sign from negative to positive as x increases through x_0, then f has a local minimum at x_0.
- If $f'(x)$ does not change sign as x increases through x_0, then f has neither a local maximum nor a local minimum at x_0.

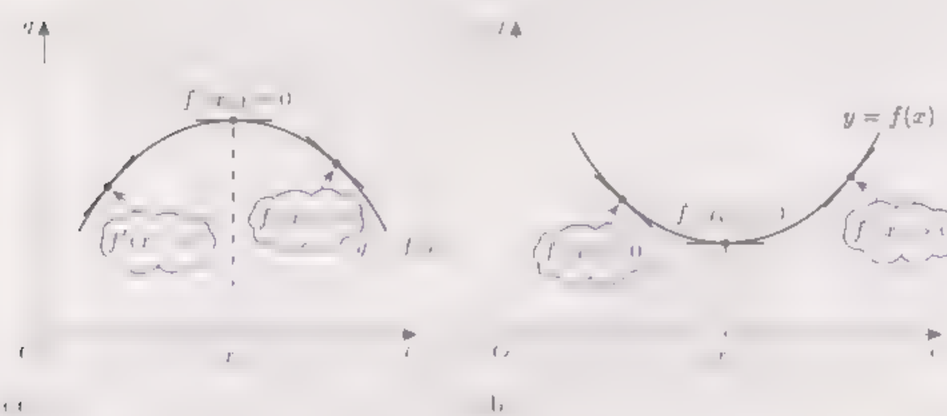


Figure 2.4 (a) Local maximum (b) Local minimum

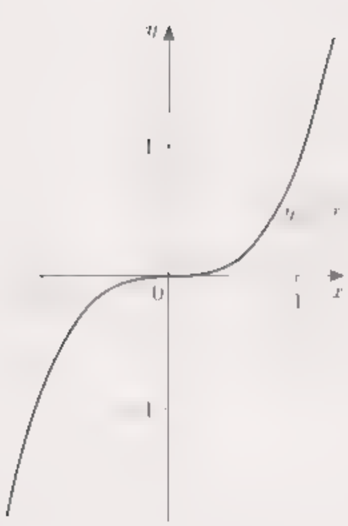


Figure 2.5 Graph of $y = x^3$

In the case of a local maximum, the function is increasing on an interval to the left of x_0 and decreasing on an interval to the right of x_0. The reverse is true for a local minimum.

Note, however, that a stationary point (at which $f'(x) = 0$) need not give rise to either a local maximum or local minimum, as demonstrated by the example of $f(x) = x^3$, whose graph is shown in Figure 2.5. This function has a stationary point at $x = 0$, which is neither a local maximum nor a local minimum. In this case the function is increasing on all intervals to the left or right of $x = 0$. This is confirmed by the sign of the derivative $f'(x) = 3x^2$, which is positive wherever $x \neq 0$.

To determine whether a general function f is increasing or decreasing to the left or right of a stationary point x_0, it is usually sufficient to check the sign of the derivative $f'(x)$ at two points, $x = x_L$ to the left of x_0 and $x = x_R$ to the right of x_0, as specified in the following strategy. The First Derivative Test can then be applied in the form given in the strategy.

Strategy

To identify the local maxima and local minima of a smooth function f.

1. Find the stationary points of f.
2. For each stationary point x_0,
   (a) choose points x_L to the left of x_0 and x_R to the right of x_0, such that the interval $[x_L, x_R]$ lies in the domain of f and there are no stationary points between x_L and x_0, nor between x_0 and x_R, and then calculate $f'(x_L)$ and $f'(x_R)$;
   (b) classify x_0 as follows:
   - if $f'(x_L) > 0$ and $f'(x_R) < 0$, then f has a local maximum at x_0;
   - if $f'(x_L) < 0$ and $f'(x_R) > 0$, then f has a local minimum at x_0;
   - if $f'(x_L)$ and $f'(x_R)$ have the same sign, then f has neither a local maximum nor a local minimum at x_0.

In this chapter we apply this strategy only to polynomial functions. Hence the interval $[x_L, x_R]$ always lies in the domain of f.

Remarks

1 For each stationary point x_0, the points x_L and x_R described in Step 2(a) are referred to as 'test points'.

2 The choice for a pair of test points is not unique. We usually choose test points near x_0 that facilitate the calculation of the derivatives in Step 2(a).

Before applying the complete strategy, here is some practice in choosing test points.

Activity 2.4 Choosing test points

In Example 2.2 it was found that the function

$$f(x) = 2x^3 + 3x^2 - 12x - 4$$

has stationary points at $x = -2$ and $x = 1$. Determine a pair of test points for each of the stationary points.

A solution is given on page 55.

The complete strategy is illustrated in the following example.

Example 2.4 Applying the First Derivative Test

Use the strategy to apply the First Derivative Test to classify each of the stationary points of the function

$$f(x) = x^3 - 3x^2 + 1.$$

Here 'classify' means 'identify as a local maximum, a local minimum, or neither'.

Solution

Step 1: The derivative of $f(x) = x^3 - 3x^2 + 1$ is

$$f'(x) = 3x^2 - 6x = 3x(x-2).$$

The equation $3x(x-2) = 0$ has solutions $x = 0$ and $x = 2$, so the stationary points of f are at $x = 0$ and $x = 2$. (These are the only stationary points of f.)

There are no stationary points between -1 and 0, nor between 0 and 1.

Step 2: To classify the stationary point at $x = 0$, we choose test points $x_L = -1$ and $x_R = 1$, say. Then we have

$$f'(x_L) = f'(-1) = 3(-1)(-3) = 9 > 0,$$
$$f'(x_R) = f'(1) = 3(1)(-1) = -3 < 0,$$

so f has a local maximum at $x = 0$, by the First Derivative Test.

To classify the stationary point at $x = 2$, we choose test points $x_L = 1$ and $x_R = 3$, say. Then we have

$$f'(x_L) = f'(1) = -3 < 0,$$
$$f'(x_R) = f'(3) = 3(3)(1) = 9 > 0,$$

so f has a local minimum at $x = 2$, by the First Derivative Test.

Activity 2.5 Applying the First Derivative Test

Use the strategy to apply the First Derivative Test to classify each of the stationary points of the function

$$f(x) = 3x^4 - 2x^3 - 9x^2 + 7,$$

noting that the derivative has a factor x.

A solution is given on page 55.

There is another test that is sometimes useful for the purpose of classifying the stationary points of a function. This involves the *second* derivative of the function.

Second derivatives

The notation f'' is pronounced as 'f double dash' or 'f double prime'.

Third, fourth and higher-order derived functions and derivatives may be defined in a similar way. When more than one order of a derivative is involved, $f'(x)$ may be called the *first* derivative (as in First Derivative Test).

When we differentiate a smooth function f, we obtain the derived function f'. Often this also is a smooth function, which can itself be differentiated. The result of doing so is the derived function of f', which is denoted by f'' and called the **second derived function** of the original function f. The value of f'' at a given point is called the **second derivative** of f at that point. (Often the term 'second derivative' is also used to refer to the second derived function f''.) For example, if $f(x) = x^2$, then

$$f'(x) = 2x \quad \text{so} \quad f''(x) = 2.$$

Since the derived function of any non-constant polynomial function is another polynomial function, results (1.4) and (2.3) can be used as before to find the second derivative of a polynomial function.

Activity 2.6 Finding a second derivative

You showed in Activity 2.1(b) that the function

$$f(x) = -x^3 + 5x^2 - 7x + 15$$

has derivative

$$f'(x) = -3x^2 + 10x - 7.$$

Find the second derivative of the function f.

A solution is given on page 55.

Second derivatives, like first derivatives, frequently crop up in mathematical models. For example, you saw in Subsection 1.3 that if $s = f(t)$ (in metres) is the position of a car moving along a straight road, measured from a fixed starting point, where t (in seconds) is the time since leaving the starting point, then the velocity of the car is represented by $v = f'(t)$, in $\mathrm{m\,s^{-1}}$. A further differentiation gives the *acceleration* of the car, $a = f''(t)$, measured in metres per second per second ($\mathrm{m\,s^{-2}}$).

The acceleration is the rate of change of velocity v with respect to time t.

The second derivative is used in a further test for classifying the stationary points of a function. To show briefly why this works, suppose that x_0 is a stationary point of a function f, so that $f'(x_0) = 0$, and suppose also that $f''(x_0) > 0$. Then there will be some open interval I, with x_0 in I, such that $f''(x) > 0$ for x in I. It follows from the Increasing/Decreasing Criterion, applied to the function f' rather than to f as originally stated, that f' is increasing on I. This means, since $f'(x_0) = 0$, that f' changes sign from negative to positive as x increases through x_0. Thus x_0 is a local minimum, by the First Derivative Test. A similar argument shows that if $f''(x_0) < 0$ at a stationary point x_0, then x_0 is a local maximum.

It is not essential to follow the details in this paragraph. If you are short of time, then move straight to the boxed statement below.

Second Derivative Test

Suppose that x_0 is a stationary point of a smooth function f; that is, $f'(x_0) = 0$.

- If $f''(x_0) < 0$, then f has a local maximum at x_0.
- If $f''(x_0) > 0$, then f has a local minimum at x_0.

Note that this test gives no information if $f''(x_0) = 0$.

One way of remembering the details of this test is indicated by the two 'faces' shown in Figure 2.6. The depiction of the 'nose' on each face, by resembling the double prime notation in f'', is a reminder that the *second* derivative is involved. The 'mouth' indicates the shape of the graph close to the stationary point x_0, while the 'eyes' give the corresponding sign of $f''(x_0)$.

Figure 2.6 'Faces' to recall the Second Derivative Test

Example 2.5 Applying the Second Derivative Test

In Example 2.4 it was found that the function

$$f(x) = x^3 - 3x^2 + 1$$

has derivative

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

and hence has stationary points at $x = 0$ and $x = 2$. Use the Second Derivative Test to classify each of these stationary points.

Solution

The second derivative of f is obtained by differentiating $f'(x)$, to obtain

$$f''(x) = 6x - 6 = 6(x - 1)$$

For the stationary point at $x = 0$, we have $f''(0) = -6 < 0$, so f has a local maximum at $x = 0$, by the Second Derivative Test.

For the stationary point at $x = 2$, we have $f''(2) = 6 > 0$, so f has a local minimum at $x = 2$, by the Second Derivative Test.

Comment

As expected, these classification results agree with those found by applying the First Derivative Test in Example 2.4.

Activity 2.7 Applying the Second Derivative Test

In Activity 2.5 you found that the function

$$f(x) = 3x^4 - 2x^3 - 9x^2 + 7$$

has derivative

$$f'(x) = 12x^3 - 6x^2 - 18x = 6x(2x - 3)(x + 1)$$

and hence has stationary points at $x = -1$, $x = 0$ and $x = \frac{3}{2}$. Use the Second Derivative Test to classify each of these stationary points.

A solution is given on page 55.

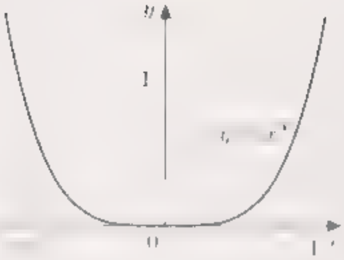


Figure 2.7 Graph of $f(x) = x^4$

The solutions to Example 2.5 and Activity 2.7 suggest that it is easier to use the Second Derivative Test than the First Derivative Test when classifying stationary points, despite the additional differentiation required. Indeed, it is frequently easier to use the Second Derivative Test for *polynomial* functions, but you should be aware that for certain functions this test cannot be applied. For example, the function $f(x) = x^4$, whose graph is shown in Figure 2.7, has a stationary point at $x = 0$ which is a local minimum. However, in this case $f'(x) = 4x^3$, so $f''(x) = 12x^2$ and hence $f''(0) = 0$. This means that the Second Derivative Test cannot be applied. On the other hand, the First Derivative Test *can* be applied. In the notation of the strategy (page 27), if $x_L = -1$ and $x_R = 1$, say, we have

$$f'(x_L) = f'(-1) = -4 < 0 \quad \text{and} \quad f'(x_R) = f'(1) = 4 > 0,$$

confirming the presence of a local minimum. For this reason, it is important to be able to use both tests.

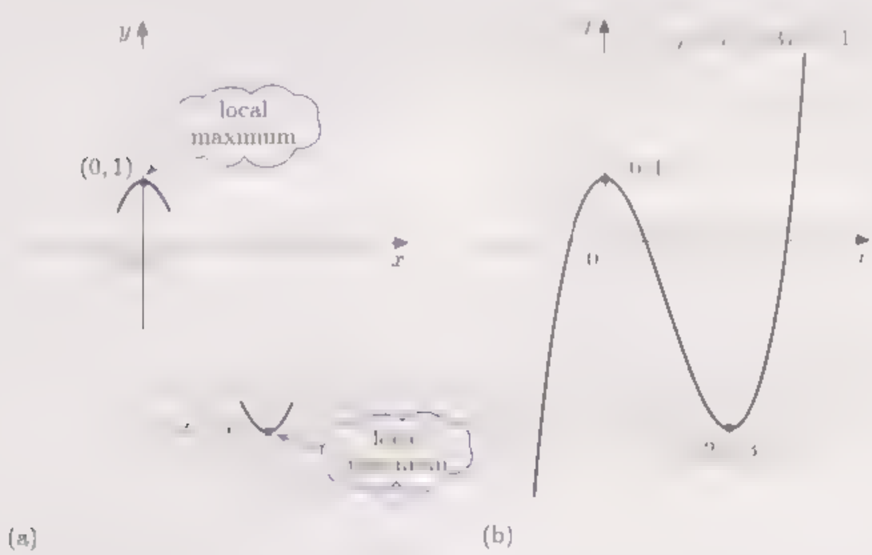
Sketching the graph

Once the stationary points of a function have been classified, it is often possible to produce a reasonable sketch of the graph of the function. For example, the function

$$f(x) = x^3 - 3x^2 + 1$$

has a local maximum at $x = 0$ and a local minimum at $x = 2$. Since we have $f(0) = 1$ and $f(2) = -3$, the shape of the graph near the points $(0, 1)$ and $(2, -3)$ can be indicated as in Figure 2.8(a). Because there are no other stationary points, f must be increasing on $(-\infty, 0)$, decreasing on $(0, 2)$ and increasing on $(2, \infty)$, so we can complete a rough sketch of the graph as shown in Figure 2.8(b). Note that for such a rough sketch it is not necessary to calculate or estimate the values of the x-intercepts, but the sketch should show the y-intercept.

The stationary points of this function were found and classified in Examples 2.4 and 2.5.



Here the y-intercept happens to correspond to a stationary point.

Figure 2.8 Sketching the graph of $y = x^3 - 3x^2 + 1$

Though rough, this sketch gives a good impression of the overall shape of the graph. In the next activity, you are invited to produce a similar rough sketch.

Activity 2.8 Sketching a graph

The stationary points of the function

$$f(x) = 3x^4 - 2x^3 - 9x^2 + 7,$$

were found and classified in Activities 2.5 and 2.7. They are at $x = -1$ (local minimum), $x = 0$ (local maximum) and $x = \frac{3}{2}$ (local minimum). Find the y-coordinates of each of these points on the graph, and hence draw a rough sketch of the graph of this function.

A solution is given on page 56.

All the functions considered as examples in this section are polynomial functions. As was pointed out earlier, all the boxed results in this subsection apply more generally, to any smooth function f. The First and Second Derivative Tests can be used to classify the stationary points of many non-polynomial functions, and the classification can be used to help sketch graphs of the functions. However, this only becomes a practical proposition once you know how to differentiate non-polynomial functions. Sections 3 and 4 will widen considerably the range of functions that you can differentiate.

2.3 Optimisation

In many modelling situations, it is required to find the greatest or least value attained by a function f as the independent variable ranges over a particular interval (arising from constraints within the model). Two such situations are illustrated in Figure 2.9, where the interval is $I = [a, b]$.

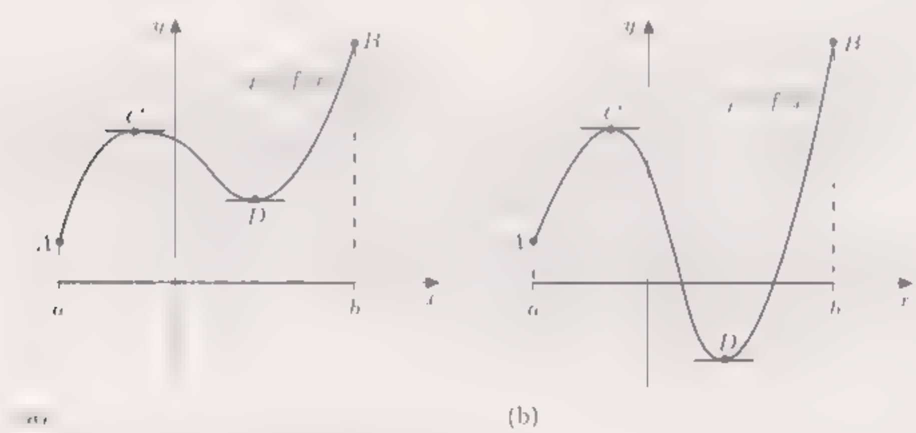


Figure 2.9 Greatest and least values

In Figure 2.9(a), the greatest value of f is attained at B and the least value is attained at A. Both A and B correspond to endpoints of the interval. In Figure 2.9(b), however, the greatest value of f is attained at an endpoint (B again) but the least value is attained at D, a local minimum.

The greatest or least values attained by a function on an interval are called the **optimum values** on that interval, and the process of finding these values is called **optimisation**. Since optimum values can occur only at endpoints or at local maxima or minima, the following procedure can be relied on to find the optimum values and where they occur.

Optimisation Procedure

To find the greatest (or least) value of a smooth function f on a closed interval I within the domain of f, proceed as follows.

1. Find the stationary points of f.
2. Evaluate f at each of the endpoints of I and at each of the stationary points inside I.
3. Choose the greatest (or least) of the function values found in Step 2.

Note that it is not essential here to classify the stationary points.

The following example illustrates the use of this procedure.

Example 2.6 Finding optimum values

Find the greatest and least values of the function

$$f(x) = -2x^3 + 3x^2 + 72x$$

on the interval $I = [-5, 5]$.

Solution

We follow the steps of the Optimisation Procedure.

Step 1: The derivative of the given function is

$$f'(x) = -6x^2 + 6x + 72 = -6(x^2 - x - 12) = -6(x + 3)(x - 4),$$

so the stationary points are at $x = -3$ and $x = 4$.

Step 2: At the endpoints of the interval $I = [-5, 5]$, the function values are

$$f(-5) = 250 + 75 - 360 = -35 \quad \text{and} \quad f(5) = -250 + 75 + 360 = 185,$$

while at the stationary points (which are both inside I) the function values are

$$f(-3) = 54 + 27 - 216 = -135 \quad \text{and} \quad f(4) = -128 + 48 + 288 = 208.$$

Step 3: The greatest value of $f(x)$ on $[-5, 5]$ is 208 (at $x = 4$), and the least value of $f(x)$ on $[-5, 5]$ is -135 (at $x = -3$).

If I had been $[-2, 3]$, then both stationary points found in Step 1 would be outside I. In this case, the greatest and least values of $f(x)$ on I are $f(3) = 189$ and $f(-2) = -116$, respectively.

Activity 2.9 Finding optimum values

In each of the following cases, find the greatest and least values of the function f on the given interval I.

(a) $f(x) = \frac{1}{3}x^3 - x^2 - 8x + 1 \qquad I = [-3, 5]$

(b) $f(x) = 3x^2 - 2x + 5 \qquad I = [-1, 4]$

Solutions are given on page 56.

Summary of Section 2

This section has introduced:

- the formula for obtaining the derivative of a polynomial function;
- the stationary points of a function f, being the points where $f'(x) = 0$, including the cases of a local maximum and a local minimum;
- the Increasing/Decreasing Criterion, which uses the sign of $f'(x)$ to determine whether f is increasing or decreasing on an interval;
- the First Derivative Test, and an associated strategy, for classifying each stationary point of a smooth function as a local maximum, a local minimum, or neither of these;
- the Second Derivative Test for classifying each stationary point of a smooth function (where applicable) as a local maximum or a local minimum;
- use of the location and classification of stationary points in sketching the graph of a smooth function;
- the Optimisation Procedure for finding the optimum (greatest or least) value of a smooth function on a closed interval.

Exercises for Section 2

Exercise 2.1

(a) Differentiate the polynomial function $f(x) = 7x^{101} - 13x^{50} + x$.

(b) What is the gradient of the graph of $y = 5x^3 + 4x^2 - 9x - 8$ at the point $(-1, 0)$?

Exercise 2.2

(a) Find the stationary points of the function

$$f(x) = -x^3 - 3x^2 + 9x + 5.$$

(b) Classify each of the stationary points found in part (a),

(i) using the strategy to apply the First Derivative Test;

(ii) using the Second Derivative Test.

(c) Find the y-coordinates of each of the stationary points on the graph, and evaluate $f(0)$. Hence draw a rough sketch of the graph of this function.

Exercise 2.3

In Activity 2.2 you found that the function

$$f(x) = x^3 - 6x^2 - 15x + 54$$

has derivative

$$f'(x) = 3x^2 - 12x - 15 = 3(x + 1)(x - 5)$$

and hence has stationary points at $x = -1$ and $x = 5$.

(a) Use the Second Derivative Test to classify each of these stationary points.

(b) Find the greatest and least values of the function f on each of the following intervals I.

(i) $I = [0, 7]$ (ii) $I = [-4, 7]$

3 Differentiating further functions

To study the latter part of Subsection 3.3 you will need a video player and the Video Tape.

In this section, the main aim is to extend the range of functions which you can differentiate. In Subsection 3.1 the derivatives of $\sin x$, $\cos x$, e^x and $\ln x$ are introduced, with some informal justification. These results are then tabulated, together with that for the derivative of x^n which you saw in Subsection 1.3.

These functions were also discussed in Chapter A3.

Subsection 3.2 states two straightforward rules that make it possible to differentiate various combinations of these basic functions. Subsection 3.3 introduces the alternative *Leibniz notation* for derivatives, which features in the Video Tape.

3.1 Basic derivatives

We look in turn at several basic functions, reviewing first a case that you have already seen.

Power functions and constant functions

It was stated in Subsection 1.3 that the derivative of any power function is given by the result

$$\text{if } f(x) = x^n, \text{ then } f'(x) = nx^{n-1}, \tag{1.4}$$

where n is any real number. You saw this formula for $f'(x)$ justified in several individual cases ($n = 0, 1, 2, 3, 4$).

In the case $n = 0$, the function is simply $f(x) = 1$, and the derivative $f'(x) = 0$ corresponds to the fact that the graph of $y = 1$ is a horizontal line, which has gradient 0 at all points. It is clear that the same applies for any other horizontal line, whose equation has the form $y = c$ where c is a constant. Any function $f(x) = c$, where c is a constant, is called a **constant function**. Thus if f is any constant function, then its derivative is zero:

$$\text{if } f(x) = c \text{ (c constant)}, \text{ then } f'(x) = 0. \tag{3.1}$$

Sine and cosine functions

There now follows an attempt to make plausible the formulas for the derivatives of the trigonometric functions sin and cos, without resorting to a formal justification.

While the following text should give you some idea of how the derivative formulas for sin and cos arise, it is not essential to follow it in order to apply these results. If you are short of time, you may prefer to note results (3.2) and (3.3) below, and then read on from that point.

The approach will be the opposite of that employed in Subsection 2.2. There information about the values or sign of $f'(x)$, obtained from its rule, was used to deduce the behaviour of the graph of $f(x)$. Here, the graph of $f(x)$ is used to sketch the graph of $f'(x)$, which suggests what the rule for $f'(x)$ might be. We start with the graph of $f(x) = \sin x$, as shown in Figure 3.1(a), overleaf.

Note that, in this graph and throughout the following discussion, x is in radians.

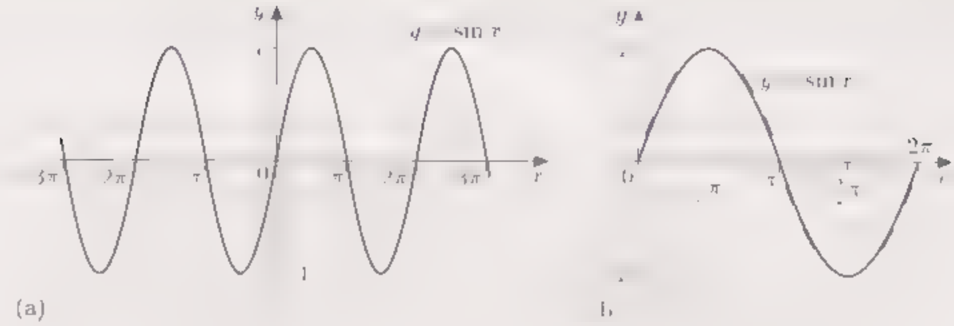


Figure 3.1 (a) Graph of $y = \sin x$ (b) Graph of $y = \sin x$ on $[0, 2\pi]$

The graph of sin is periodic, with period 2π, so the same will be true of its derivative, which gives the gradient at each point of the graph. It suffices therefore to consider the portion of the graph in the interval $[0, 2\pi]$, as shown in Figure 3.1(b). Within this interval, sin has stationary points (zero gradient on the graph) at $x = \frac{1}{2}\pi$ and $x = \frac{3}{2}\pi$, so we must have

$$f'(\tfrac{1}{2}\pi) = f'(\tfrac{3}{2}\pi) = 0.$$

The graph of sin has positive gradient on the intervals $(0, \frac{1}{2}\pi)$ and $(\frac{3}{2}\pi, 2\pi)$, and negative gradient on the interval $(\frac{1}{2}\pi, \frac{3}{2}\pi)$, so

$$f'(x) > 0 \text{ for } x \text{ in } (0, \tfrac{1}{2}\pi) \text{ and for } x \text{ in } (\tfrac{3}{2}\pi, 2\pi),$$
$$f'(x) < 0 \text{ for } x \text{ in } (\tfrac{1}{2}\pi, \tfrac{3}{2}\pi).$$

This assists in sketching the graph of $f'(x)$, since this graph is above the x-axis where $f'(x) > 0$ and below the x-axis where $f'(x) < 0$. By looking at the gradient in Figure 3.1(b), we see that the derived function f' is decreasing on the interval $(0, \pi)$ and increasing on the interval $(\pi, 2\pi)$.

Finally, we ask what are the greatest and least values that the gradient of sin can take within the interval $[0, 2\pi]$. The greatest value occurs where the graph is at its steepest increase, which seems to be at $x = 0$ and $x = 2\pi$. The least value occurs where the graph is at its steepest decrease, which seems to be at $x = \pi$. The symmetry of the graph suggests that the gradient at $x = \pi$ is the negative of the gradient at $x = 0$.

The following activity indicates that the gradient at $x = 0$ of the graph of sin is close to 1.

Activity 3.1 Calculating the gradient of sin at x = 0

According to equation (1.3), the derivative of $f(x) = \sin x$ at $x = 0$ is

$$f'(0) = \lim_{h \to 0} \left(\frac{\sin(0 + h) - \sin 0}{h} \right) = \lim_{h \to 0} \left(\frac{\sin h}{h} \right).$$

Remember that your calculator needs to be in radian mode for this calculation.

Calculate the values of the ratio $(\sin h)/h$ for each of $h = 0.1, 0.01, 0.001$, giving your answers to six decimal places. To what limit does the ratio seem to tend as $h \to 0$?

A solution is given on page 56.

The result of Activity 3.1 and the text beforehand suggest that if $f(x) = \sin x$, then

$$f'(0) = 1, \quad f'(\pi) = -1, \quad f'(2\pi) = 1$$

Adding this information to that obtained previously, we can sketch the graph of $f'(x)$ as shown in Figure 3.2.

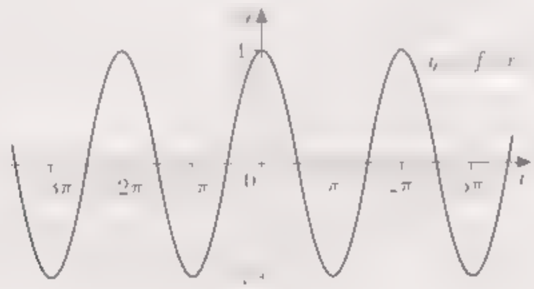


Figure 3.2 Sketch graph of $y = f'(x)$, where $f(x) = \sin x$

This looks like the graph of the cosine function! In fact, the derivative of sin is indeed cos; that is,

$$\text{if } f(x) = \sin x, \text{ then } f'(x) = \cos x. \tag{3.2}$$

Note that result (3.2) holds only when x is in radians. The same applies to result (3.3) below.

We could undertake a similar argument to make plausible the formula for the derivative of $f(x) = \cos x$, but there is a quicker approach, based on result (3.2).

The graph of cos (derivative of sin) is shown in Figure 3.2. This graph is the same *shape* as that of sin, but translated by $\frac{1}{2}\pi$ to the left. Hence the graph of the derivative (giving the gradient) of $f(x) = \cos x$ has the same shape once again, but translated a further $\frac{1}{2}\pi$ to the left. However, the outcome of translating the graph of $\cos x$ in this way is the graph of $-\sin x$. This explains the result that

$$\text{if } f(x) = \cos x, \text{ then } f'(x) = -\sin x. \tag{3.3}$$

Translating the graph of $\cos x$ to the left by $\frac{1}{2}\pi$ gives the graph of $\cos(x + \frac{1}{2}\pi)$, as explained in Chapter A3. Using trigonometric identities from Chapter A2, we have $\cos(x + \frac{1}{2}\pi) = \cos(\frac{1}{2}\pi - (-x)) = \sin(-x) = -\sin x$.

Exponential and logarithmic functions

First recall the shape of graphs of the form $y = a^x$ (a constant), some instances of which are shown in Figure 3.3.

As with the trigonometric functions earlier, it is not essential to follow the explanatory details here of how the derivative formulas for exp and ln arise. If you are short of time, you may prefer to note results (3.5) and (3.6) below, and then read on from that point.

Exponential functions were introduced in Chapter A3, Subsection 3.2.

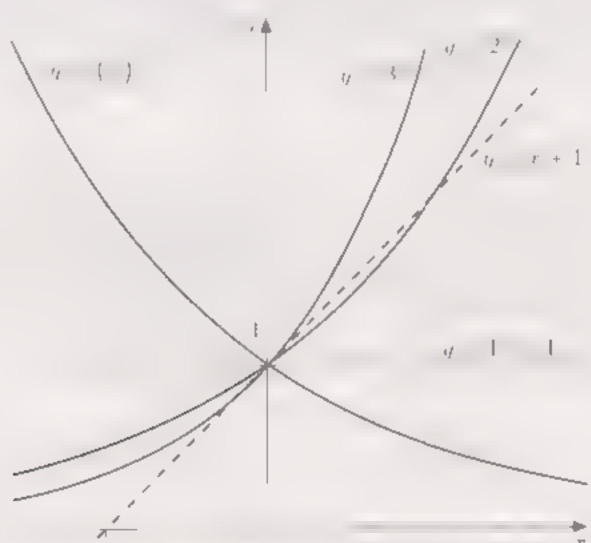


Figure 3.3 Graphs of $y = a^x$ for $a = \frac{1}{2}, 1, 2, 3$

Each of the graphs in Figure 3.3 passes through the point $(0,1)$, since $a^0 = 1$ whenever $a \neq 0$. It seems that the greater the value of a, the larger is the gradient of the graph of $y = a^x$ at $(0,1)$. For $y = 2^x$, the gradient at $(0,1)$ appears to be less than 1, whereas for $y = 3^x$ the gradient at this point appears to be greater than 1. This can be seen by comparison with the broken line through $(0,1)$ which represents $y = x + 1$ and has gradient 1. The number $e = 2.718\ldots$ has the property that the gradient of the graph of $y = e^x$ at the point $(0,1)$ is exactly 1, as illustrated in Figure 3.4.

The value of e stored by your calculator can be found by evaluating $e^1 = \exp 1$.

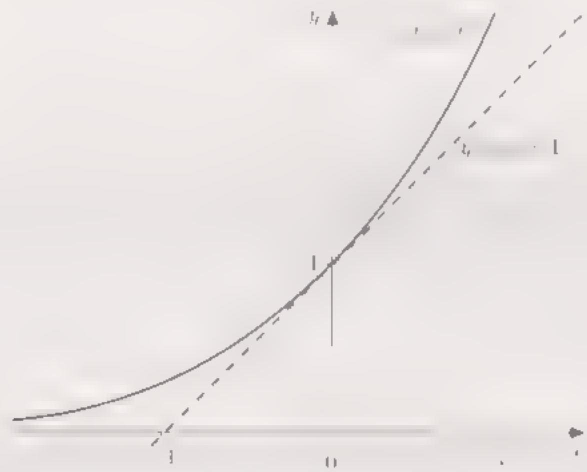


Figure 3.4 Graph of $y = e^x$

This defining feature of the number e was stated in Chapter A3. In terms of derivatives, it means that if $f(x) = e^x$, then $f'(0) = 1$. According to equation (1.3), this is the same as saying that

$$\lim_{h \to 0}\left(\frac{e^{0+h} - e^0}{h}\right) = 1; \quad \text{that is,} \quad \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = 1. \tag{3.4}$$

The next activity asks you to check that this is the case.

Activity 3.2 Checking equation (3.4)

Calculate the values of the ratio $(e^h - 1)/h$ for each of $h = 0.1, 0.001, 0.000\,01$, giving your answers to six decimal places. To what limit does the ratio seem to tend as $h \to 0$?

A solution is given on page 56.

It is now straightforward to find the derivative of $f(x) = e^x$ from definition (1.3). Thus we have

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{e^{x+h} - e^x}{h} \\ &= \frac{e^x e^h - e^x}{h} \quad \text{(by a rule for powers)} \\ &= e^x\left(\frac{e^h - 1}{h}\right)\end{aligned}$$

On taking the limit as $h \to 0$, we obtain

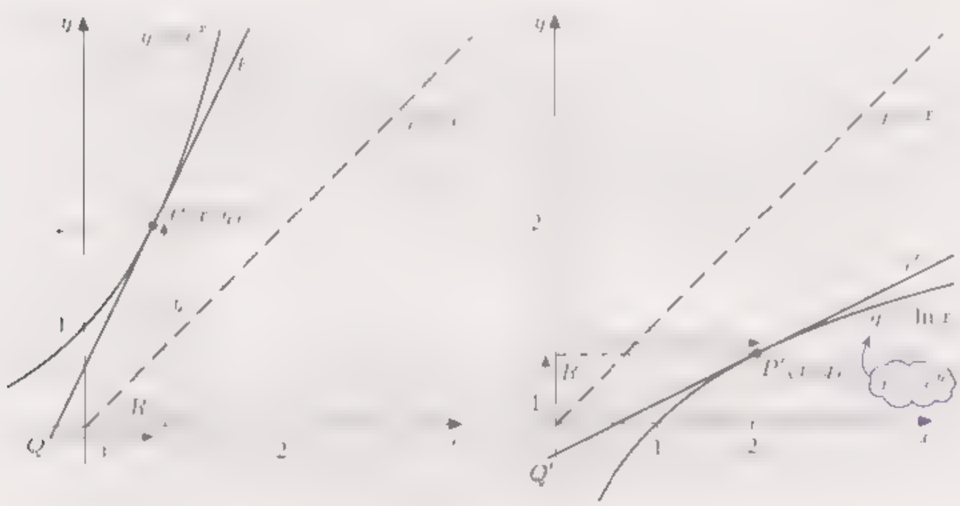
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) \\ &= e^x \lim_{h\to 0}\left(\frac{e^h-1}{h}\right) \\ &= e^x \quad \text{(by equation (3.4))}. \end{aligned}$$

In other words,

$$\text{if } f(x) = e^x, \text{ then } f'(x) = e^x. \tag{3.5}$$

This result states that the derivative of the exponential function is the exponential function!

The following illuminating geometrical interpretation of this result leads on to the formula for the derivative of the function $\ln x$. Suppose that $P(x, y)$ is any point on the graph of $y = e^x$, and that ℓ is the tangent to the curve at P; see Figure 3.5(a). By result (3.5), the line ℓ has gradient e^x, which is also the ratio 'rise over run' for the triangle PQR that has its base QR on the x-axis. Since the 'rise' RP is $y = e^x$, the 'run' QR must be 1. In other words, any tangent at (x, y) to the graph of $y = e^x$ intersects the x-axis one unit to the left, at $(x - 1, 0)$.



(a) (b)

Figure 3.5 (a) Tangent ℓ to graph of $y = e^x$ (b) Corresponding tangent ℓ' to graph of $y = \ln x$

This geometrical result helps us to deduce the derivative of the natural logarithm function ln which, as you may recall, is the inverse of the exponential function exp. The graph of an inverse function is obtained from that of the original function by reflection in the 45° line, with equation $y = x$.

The function ln was introduced in Chapter A3, Subsection 4.3. Section 4 of that chapter also dealt more widely with inverse functions.

Suppose then that the graph and lines in Figure 3.5(a) are reflected in the 45° line, accompanied by a swapping of the labels x and y throughout, so we again have a horizontal x-axis and vertical y-axis. The outcome is shown in Figure 3.5(b), where P', Q', R' are respectively the images of the points P, Q, R, and ℓ' is the image of the line ℓ. The reflected graph has equation $x = e^y$ or, equivalently, $y = \ln x$.

Now the line ℓ' is tangent to this graph at P', and the gradient of ℓ', with reference to the triangle $P'Q'R'$, equals the 'rise' $Q'R' = 1$ divided by the 'run' $R'P' = x$, that is, $1/x$. We have therefore arrived at the following result:

$$\text{if } f(x) = \ln x \ (x > 0), \text{ then } f'(x) = \frac{1}{x} \ (x > 0). \tag{3.6}$$

Although you may refer to the Handbook for these derivatives, they crop up so regularly in calculus that it is probably worth trying to remember them.

The differentiation results encountered in this subsection are collected together in the table below.

Table 3.1

Function $f(x)$	Derivative $f'(x)$
c	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x \quad (x > 0)$	$1/x \quad (x > 0)$

3.2 Sum and Constant Multiple Rules

These rules are straightforward consequences of definition (1.3) of the derivative, though the proofs are not given here.

The derivatives of some basic functions are given in Table 3.1. Sums and constant multiples of these basic functions can be differentiated using the following rules.

Sum Rule

If k is a function with rule of the form $k(x) = f(x) + g(x)$, where f and g are smooth functions, then k is smooth and

$$k'(x) = f'(x) + g'(x).$$

For example, if $f(x) = x^2$ and $g(x) = \sin x$, then (from Table 3.1)

$$f'(x) = 2x \quad \text{and} \quad g'(x) = \cos x,$$

so the function $k(x) = x^2 + \sin x$ has derivative

$$k'(x) = 2x + \cos x.$$

Constant Multiple Rule

If k is a function with rule of the form $k(x) = cf(x)$, where f is a smooth function and c is a constant, then k is smooth and

$$k'(x) = cf'(x).$$

For example, if $f(x) = x^2$ and $c = 5$, then the function $k(x) = 5x^2$ has derivative

$$k'(x) = 5 \times 2x = 10x.$$

These two rules are often used in combination. For example, if $f(x) = 7x^2 - 4\sin x$, then we can write immediately (using these rules) that

$$f'(x) = 7(2x) + (-4)(\cos x) = 14x - 4\cos x.$$

In fact, these two rules were in effect taken for granted earlier:

◇ in Frame 4 of the audio tape, in giving the gradient formula for a general quadratic function, repeated in result (1.1); See Subsection 1.2

◇ in giving the formula for the derivative of any polynomial function, in the form of result (2.2) or (2.3). See Subsection 2.[illegible]

In future they will be used without comment.

Activity 3.3 Using the Sum and Constant Multiple Rules

Differentiate each of the following functions.

(a) $f(x) = x^4 - 2\sin x$ (b) $g(x) = 3e^x - 4\ln x$

(c) $p(t) = \dfrac{6}{t^2} + 3\cos t$

Solutions are given on page 57.

3.3 Other notations for derivatives

In Subsection 1.3 you saw the definition of differentiation of a smooth function f as the process of finding the derived function f' of f. There are, however, other notations in common use for describing derivatives. Which notation to use depends partly on the context, as you will see.

The function notation f' is the preferred notation of those who study calculus principally as a branch of mathematics. It arises from the view that the fundamental objects within calculus are functions rather than variables. When calculus was developed, however, in the seventeenth century, today's notion of 'function' did not exist. Newton and Leibniz were concerned with geometric problems (for example, finding tangents to curves), so it was natural for them to work with the variables involved in the equations of curves.

The name of Leibniz is pronounced 'lie-b-nitz'.

The notation invented by Leibniz can be used for any pair of related variables, and it makes explicit mention of these variables. For example, if the variables y and x are related by the equation

$$y = x^2,$$

then Leibniz would have spoken of 'the derivative of y with respect to x' and written this as

$$\frac{dy}{dx} = 2x.$$

Here dy/dx has the same meaning as $f'(x)$, where f is the function $f(x) = x^2$. Note that the expression dy/dx should be regarded as a complete symbol since, despite its appearance, it *is not a fraction* in the ordinary sense. In particular, the two ds which appear in it *do not cancel*. Nor do the subexpressions dy and dx have any individual meaning.

The derivative on the left-hand side may also be written as dy/dx. It is pronounced as 'dee-y by dee-x' or sometimes just as 'dee-y dee-x'.

When the derivative for a particular value of the independent variable is required, the Leibniz notation has to be adapted. For example, if $f(x) = x^2$ then we have $f'(3) = 6$. In Leibniz notation, with $y = x^2$, this value of the derivative at $x = 3$ would be expressed as

$$\left.\frac{dy}{dx}\right|_{x=3} = 6$$

Leibniz arrived at his notation by thinking of the derivative as a means of describing the 'rate of change' of a dependent variable as the independent variable changes. This may be visualised by starting from a point (x, y) on the graph of $y = f(x)$, and then making a small change δx in x which leads to a small change δy in y; see Figure 3.6.

The Greek letter δ (delta) is placed in front of a variable to indicate a small change in that variable. Symbols such as δx and δy are 'complete symbols', not products.

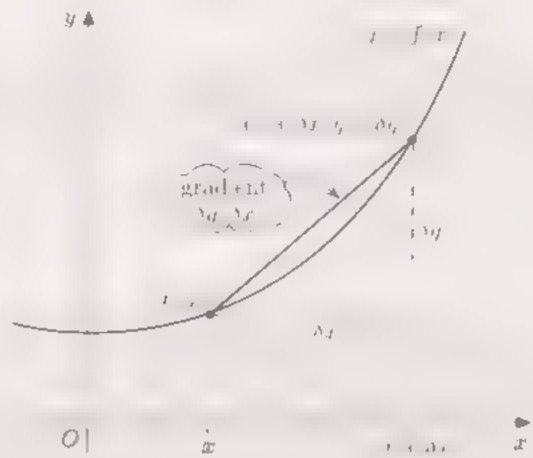


Figure 3.6 Small change δx in x and corresponding small change δy in y

The ratio $\delta y/\delta x$ is the 'average' rate of change of y with respect to x over the interval $[x, x + \delta x]$. Leibniz used the notation dy/dx to indicate the limiting value to which the quotient $\delta y/\delta x$ approaches as δx becomes small; that is,

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

This matches the definition of derivative in equation (1.3), with h replaced by δx and $f(x + h) - f(x)$ replaced by δy.

It is then natural to regard the derivative dy/dx as the 'instantaneous' rate of change of y with respect to x. While dy/dx is not itself a quotient, it is the limit of quotients of the form $\delta y/\delta x$ as $\delta x \to 0$.

As mentioned earlier, the Leibniz notation can be used whenever two variables are related. For example, the equation

$$A = \pi r^2$$

relates the area A of a circle to its radius r. The derivative of A with respect to r is

$$\frac{dA}{dr} = 2\pi r.$$

This represents the instantaneous rate of change of A with respect to r.

As another example, if a car moving along a straight road has position $s = 30t - t^2$ metres from its starting point after t seconds, then its velocity v (in $\mathrm{m\,s^{-1}}$) at time t is given by

This is the situation which was considered in Example 1.2.

$$v = \frac{ds}{dt} = 30 - 2t.$$

For this example, a further differentiation with respect to t gives the acceleration a of the car, which we originally denoted by $a = f''(t)$ (where $s = f(t)$). In Leibniz notation, this is written as

See Subsection 2.2, in the text immediately after Activity 2.6.

$$a = \frac{d^2 s}{dt^2} = -2.$$

The symbol $\dfrac{d^2 s}{dt^2}$ is pronounced as 'dee-two-s by dee-t-squared'.

In general, if $y = f(x)$, the function notation and Leibniz notation are matched by the statements

$$\frac{dy}{dx} = f'(x) \quad \text{and} \quad \frac{d^2y}{dx^2} = f''(x).$$

If we have (say) $s = 30t - t^2$, a variation of the Leibniz notation for the derivative is to write

$$\frac{d}{dt}(s) \quad \text{or} \quad \frac{d}{dt}(30t - t^2).$$

The notation $\frac{d}{dt}(\quad)$ is pronounced as 'dee by dee t of ( )'.

The latter form enables us to write derivative results very concisely. For example, two of the derivatives from Table 3.1 can be expressed as

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

The Leibniz notation is often used in applied mathematics, science and engineering, where there tends to be greater emphasis on variables than on functions. The variables frequently represent quantities with a physical interpretation, which are related to one another within a mathematical model.

One other notation is worth mention here. In textbooks on motion, especially in physics and engineering, differentiation with respect to *time* t is often represented by a dot above the dependent variable, as follows:

$$\dot{s} = \frac{ds}{dt}.$$

The expression $\dot{s}$ is pronounced as 's dot'.

This notation was devised by Newton, who used calculus to study motion. The corresponding second derivative is denoted by

$$\ddot{s} = \frac{d^2s}{dt^2}.$$

The expression $\ddot{s}$ is pronounced as 's double dot'.

Activity 3.4 Using other notations

Find each of the following derivatives, using the notation indicated.

(a) $\dfrac{dV}{dr}$ and $\dfrac{d^2V}{dr^2}$, where $V = \frac{4}{3}\pi r^3$

(b) $f'(5)$ and $f''(3)$, where $f(x) = 45 \ln x$

(c) $\left.\dfrac{ds}{dt}\right|_{t=2}$, where $s = 10 - 5t^2$

(d) $\dot{u}$ and $\ddot{u}$, where $u = 3 \cos t$

Solutions are given on page 57.

This section concludes with a video band that provides a further explanation of most of the results in Table 3.1, page 40. This video band, which uses the notation $\exp x$ rather than e^x for the exponential function, includes animation of tangents to curves and use of a hypothetical device called a 'gradient meter'.

Now watch Video Band C(ii), 'Visualising differentiation'.

Summary of Section 3

This section has introduced:

- the formulas for the derivatives of some basic functions, as listed in Table 3.1 on page 40;
- the Sum and Constant Multiple Rules for differentiating combinations of smooth functions;
- the Leibniz notation and Newton's notation for derivatives.

Exercises for Section 3

Exercise 3.1

Differentiate each of the following functions.

(a) $u(x) = 4\sin x - 5\cos x$ (b) $v(x) = 7\ln x - 2e^x$

(c) $w(x) = 3x^2 - 2x + 1 - \dfrac{1}{x} + \dfrac{2}{x^2}$

Exercise 3.2

Find each of the following derivatives, using the notation indicated.

(a) $\dfrac{dP}{du}$ and $\dfrac{d^2P}{du^2}$, where $P = \cos u - \sin u$

(b) $\dot{v}$ and $\ddot{v}$, where $v = -e^t + 17t + 3$

(c) $\left.\dfrac{dw}{ds}\right|_{s=3}$ where $w = 5s^4 - 10s^2$

4 Products, quotients and composites

In Subsection 3.1 you saw the derivatives of various basic functions, and in Subsection 3.2 you met the Sum Rule and Constant Multiple Rule for differentiating sums or constant multiples of smooth functions. By using Table 3.1 and these two results you can differentiate many functions, including all polynomial functions.

These derivatives are given in Table 3.1 on page 40.

In this section you will see how to differentiate various types of function whose rules have a more complex structure. The three subsections deal in turn with functions that can be viewed as products, quotients or *composites* of a pair of more basic functions. At the end of Subsection 4.3, a table of derivatives is given which is a more general version of Table 3.1.

4.1 Product Rule

Here is an example of a function that cannot be differentiated using the Sum and Constant Multiple Rules:

$$k(x) = x^2 \sin x. \tag{4.1}$$

This function, $k(x) = x^2 \times \sin x$, is the *product* of the two basic functions $f(x) = x^2$ and $g(x) = \sin x$, each of which you already know how to differentiate. In order to differentiate k, therefore, we need a 'product rule' for differentiation. Such a rule is available, and is stated below.

Product Rule

If k is a function with rule of the form $k(x) = f(x)g(x)$, where f and g are smooth functions, then k is smooth and

$$k'(x) = f'(x)g(x) + f(x)g'(x). \tag{4.2}$$

This rule and the others quoted in this section are consequences of definition (1.3) of the derivative, though the proofs are not given here.

The right-hand side of equation (4.2) is the sum of two terms, each of which is the product of one of the constituent functions, f or g, with the derivative of the other.

For example, if $k(x) = x^2 \sin x$, as in equation (4.1), then $k(x) = f(x)g(x)$, where $f(x) = x^2$ and $g(x) = \sin x$. Now $f'(x) = 2x$ and $g'(x) = \cos x$ so, by the Product Rule, we have

$$\begin{aligned} k'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= (2x)(\sin x) + (x^2)(\cos x) \\ &= 2x \sin x + x^2 \cos x. \end{aligned}$$

Here are some similar examples for you to try.

Activity 4.1 Using the Product Rule

By first identifying each of the following functions as a product of the form $f(x)g(x)$, and then using the Product Rule, differentiate each function.

(a) $k(x) = xe^x$ (b) $k(x) = (2x^2 - 1)\cos x$ (c) $k(x) = \sqrt{x}(x + \sin x)$

Solutions are given on page 57.

With practice, it is possible to apply the Product Rule informally, without assigning names to the two constituent functions, by remembering in words that the derivative of 'product = (first)×(second)' is given by

$$\begin{pmatrix}\text{derivative}\\ \text{of product}\end{pmatrix} = \begin{pmatrix}\text{derivative}\\ \text{of first}\end{pmatrix} \times (\text{second}) + (\text{first}) \times \begin{pmatrix}\text{derivative}\\ \text{of second}\end{pmatrix}.$$

You should aim to be able to differentiate a function which is a product in this informal manner because it is often inconvenient to give the names f and g to the constituent functions, as the following activity demonstrates.

Activity 4.2 Using the Product Rule informally

Differentiate each of the following functions.

(a) $f(x) = (3x^2 + 2x + 1)e^x$ (b) $g(x) = \sin x \cos x$

(c) $f(t) = (t^2 + 3)\ln t$

Solutions are given on page 57.

In Subsection 3.3 we described the alternative Leibniz notation for derivatives. In terms of this notation, the Product Rule may be expressed as follows.

Both this and the earlier form of the Product Rule can be written in different ways. For example, the right-hand side here can also be written as

$$u\frac{dv}{dx} + v\frac{du}{dx}.$$

Product Rule (Leibniz form)

If $y = uv$, where $u = f(x)$ and $v = g(x)$, then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

To compare this equation with equation (4.2), it is necessary to let $y = k(x)$.

4.2 Quotient Rule

Now consider again the basic functions $f(x) = x^2$ and $g(x) = \sin x$. The function

$$k(x) = \frac{f(x)}{g(x)} = \frac{x^2}{\sin x} \qquad (4.3)$$

cannot be differentiated using the Sum, Constant Multiple and Product Rules. In this case $k(x)$ is the *quotient* of $f(x)$ and $g(x)$; the function k can be differentiated using the following rule.

There is no danger of dividing by 0 here since any points x for which $g(x) = 0$ are not in the domain of k.

Quotient Rule

If k is a function with rule of the form $k(x) = f(x)/g(x)$, where f and g are smooth functions, then k is smooth and

$$k'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}. \qquad (4.4)$$

The numerator of the right-hand side of equation (4.4) is the difference of two terms, each of which is the product of one of the constituent functions, f or g, with the derivative of the other.

For example, if $k(x) = x^2/\sin x$, as in equation (4.3), then $k(x) = f(x)/g(x)$, where $f(x) = x^2$ and $g(x) = \sin x$. Now $f'(x) = 2x$ and $g'(x) = \cos x$ so, by the Quotient Rule, we have

$$\begin{aligned} k'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{(\sin x)(2x) - (x^2)(\cos x)}{(\sin x)^2} \\ &= \frac{2x\sin x - x^2\cos x}{\sin^2 x} \end{aligned}$$

Here are some similar examples for you to try.

Activity 4.3 Using the Quotient Rule

By first identifying each of the following functions as a quotient of the form $f(x)/g(x)$, and then using the Quotient Rule, differentiate each function.

(a) $k(x) = \dfrac{e^x}{x}$ (b) $k(x) = \dfrac{\cos x}{x^2 - 1}$ (c) $k(x) = \dfrac{x+1}{x-1}$

Solutions are given on page 57.

As with the Product Rule, you should aim to apply the Quotient Rule without assigning names to the constituent functions, by remembering in words that the derivative of 'quotient = (top)/(bottom)' is given by

$$\begin{pmatrix}\text{derivative}\\ \text{of quotient}\end{pmatrix} = \frac{\text{bottom} \times \begin{pmatrix}\text{derivative}\\ \text{of top}\end{pmatrix} - (\text{top}) \times \begin{pmatrix}\text{derivative}\\ \text{of bottom}\end{pmatrix}}{(\text{bottom})^2}$$

Activity 4.4 Using the Quotient Rule informally

Differentiate each of the following functions.

(a) $f(x) = \dfrac{\sin x}{\cos x}$ (b) $g(u) = \dfrac{\ln u}{u^2 + 3}$ [illegible]

Solutions are given on page 58.

Comment

Since $\tan x = \sin x/\cos x$, the result of part (a) shows that the derivative of $f(x) = \tan x$ is $f'(x) = \sec^2 x$.

In Leibniz notation, the Quotient Rule may be expressed as follows.

> **Quotient Rule (Leibniz form)**
>
> If $y = u/v$, where $u = f(x)$ and $v = g(x) \neq 0$, then
>
> $$\frac{dy}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right).$$

As with the Product Rule, the right hand side here can be written in other ways, for example, as

$$\frac{1}{v^2}\left(\frac{du}{dx}v - u\frac{dv}{dx}\right)$$

4.3 Composite Rule

In the previous two subsections, we used the basic functions $f(x) = x^2$ and $g(x) = \sin x$ to construct the product $f(x)g(x) = x^2 \sin x$ and the quotient $f(x)/g(x) = x^2/\sin x$. There is yet another way in which these two basic functions can be combined to form a different type of function.

If we *first* apply the function f to x, to obtain $f(x)$, and *then* apply the function g to $f(x)$, we obtain the function

$$k(x) = g(f(x)) = \sin(x^2). \tag{4.5}$$

The composite function $k(x) = g(f(x))$ is defined only if any possible output from f is an allowable input for g. In the language of Chapter A3, Section 1, the image set of f must be contained within the domain of g.

A function k with rule of the form $k(x) = g(f(x))$ is called a **composite function**, or informally a 'function of a function'. The expression $g(f(x))$ can be thought of as

$$g(u), \text{ where } u = f(x).$$

The intermediate variable u labels the outputs from the 'inner' function f, which are also the inputs to the 'outer' function g. The rule for g in the case of equation (4.5) can be expressed as $g(u) = \sin u$.

Some other examples of composite functions are:

$$k(x) = e^{2x}, \quad \text{where } u = f(x) = 2x \text{ and } g(u) = e^u;$$
$$k(x) = \cos^2 x, \quad \text{where } u = f(x) = \cos x \text{ and } g(u) = u^2.$$

Note that $\cos^2 x$ means $(\cos x)^2$.

Activity 4.5 Recognising composite functions

Identify each of the following as a composite function of the form $k(x) = g(f(x))$, by specifying the rules for the 'inner' function $u = f(x)$ and for the 'outer' function $g(u)$.

(a) $k(x) = \ln(x^2 + 1)$ (b) $k(x) = \cos(x^3)$ (c) $k(x) = \cos^3 x$

(d) $k(x) = e^{\sin x}$

Solutions are given on page 58.

To differentiate composite functions, we use the following rule.

Composite Rule

If k is a function with rule of the form $k(x) = g(f(x))$, where f and g are smooth functions, then k is smooth and

$$k'(x) = g'(f(x))f'(x). \tag{4.6}$$

This rule needs to be applied carefully. The right-hand side of equation (4.6) is a product of two derivatives, but the first of these involves the function g' evaluated at $f(x)$. When first learning how to use the Composite Rule, it is helpful to replace $f(x)$ by the variable u, as in the text about composite functions above.

For example, if $k(x) = \sin(x^2)$, as in equation (4.5), then $k(x) = g(f(x))$, where $u = f(x) = x^2$ and $g(u) = \sin u$.

Now $f'(x) = 2x$ and $g'(u) = \cos u$ so, by the Composite Rule, we have

$$\begin{aligned} k'(x) &= g'(f(x))f'(x) \\ &= g'(u)f'(x) \quad (\text{where } u = f(x) = x^2) \\ &= (\cos u)(2x) \\ &= \cos(x^2) \times 2x \quad (\text{since } u = x^2) \\ &= 2x\cos(x^2). \end{aligned}$$

Here are some similar examples for you to try.

Activity 4.6 Using the Composite Rule

By first identifying each of the following functions as a composite of the form $k(x) = g(f(x))$, and then using the Composite Rule, differentiate each function.

(a) $k(x) = e^{x^3}$ (b) $k(x) = \sin(2x)$ (c) $k(x) = \cos(e^x)$

Solutions are given on page 58.

Once again, you should aim to apply the Composite Rule without assigning names to the constituent functions, by remembering in words that the derivative of 'composite = outer(inner)' is given by

$$\begin{pmatrix}\text{derivative}\\ \text{of composite}\end{pmatrix} = \begin{pmatrix}\text{derivative}\\ \text{of outer}\end{pmatrix} \times \begin{pmatrix}\text{derivative}\\ \text{of inner}\end{pmatrix}.$$

If $k(x) = g(f(x))$, then f is the inner function and g the outer function.

However, you must also remember that the derivative of the outer function g is evaluated at $f(x)$, which means that the 'words' version above is not self-contained in the manner of the corresponding statements for the Product and Quotient Rules. Although you should aim eventually to be able to differentiate a function which is a composite in this informal manner, mastery may take longer than with the earlier rules.

You should explicitly assign names to the constituent functions for as long as you find this helpful.

Activity 4.7 Using the Composite Rule informally

Differentiate each of the following functions.

(a) $f(x) = \cos(4x)$ (b) $g(x) = \ln(x^2 + x + 1)$ (c) $v(t) = \sin(\cos t)$

Solutions are given on page 58.

The Leibniz form of the Composite Rule is particularly memorable. When stated in this form, the result is usually referred to as the *Chain Rule*.

Chain Rule (Leibniz form of Composite Rule)

If $y = g(u)$, where $u = f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

This version has the merit (for memorising its form) but also the danger of looking like an obvious statement about fractions (which it is *not*!).

For example, if $y = \sin(x^2)$, then $y = \sin u$, where $u = x^2$, so

$$\frac{dy}{du} = \cos u, \quad \frac{du}{dx} = 2x \quad \text{and}$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(2x) = 2x\cos(x^2).$$

You have already differentiated specific functions of this type: $\sin(2x)$ in Activity 4.6(b), and $\cos(4x)$ in Activity 4.7(a).

Finally, note that the Composite Rule enables us to generalise somewhat the results of Table 3.1, to cope with functions such as $\sin(ax)$, $\cos(ax)$, e^{ax} and $\ln(ax)$, where a is any non-zero constant. For suppose that the derivative of the function g is known, and that the function k is defined by $k(x) = g(ax)$. This has the form of a composite function $k(x) = g(f(x))$, for which the inner function is $f(x) = ax$, with derivative $f'(x) = a$. It follows from the Composite Rule that

$$k'(x) = g'(f(x))f'(x) = ag'(ax).$$

By applying this result to the functions in the last four lines of Table 3.1, we obtain the amended table below

This table, in which a is a non-zero constant, appears in the Handbook. Note that the results of Activities 4.6(b) and 4.7(a) are consistent with entries in this table.

Table 4.1

Function $f(x)$	Derivative $f'(x)$
c	0
x^n	nx^{n-1}
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
e^{ax}	ae^{ax}
$\ln(ax)$ $(ax > 0)$	$1/x$ $(ax > 0)$

Summary of Section 4

This section has introduced:

- the Product Rule, Quotient Rule and Composite Rule (Chain Rule) for differentiating functions which are appropriate combinations of other more basic functions;
- the formulas for the derivatives of some basic functions of the form $f(x) = g(ax)$, where a is a constant, as listed in Table 4.1 above.

Exercises for Section 4

Exercise 4.1

Use the Product Rule to differentiate each of the following functions.

(a) $f(x) = (x^3 + x)\sin x$ (b) $g(x) = e^x \cos x$ (c) $u(t) = \sqrt{t}\ln t$

Exercise 4.2

Use the Quotient Rule to differentiate each of the following functions.

(a) $g(x) = \dfrac{x^4}{\ln x}$ (b) $k(t) = \dfrac{\sin t - t^2}{e^t}$ (c) $f(x) = \dfrac{1}{\cos x}$

Exercise 4.3

Use the Composite Rule to differentiate each of the following functions.

(a) $k(x) = \sin(\ln x)$ (b) $k(x) = \sin^4 x$ (c) $k(x) = e^{\sqrt{x}}$

(d) $p(t) = \ln(e^t + t)$ (e) $q(s) = \cos\left(\dfrac{1}{s^2}\right)$ (f) $r(y) = \dfrac{1}{\cos^2 y}$

5 Optimisation with the computer

To study this section you will need access to your computer, together with Computer Book C.

In this section you will see that the computer can be used to find the derivative of any smooth function whose rule is entered. This means that, even where the function concerned is too complicated to permit the general differentiation results in this chapter to be applied with ease 'by hand', an answer can still be obtained from the computer. Equally, the computer can be used to provide an independent check on any calculation done by hand.

You are expected to be able to differentiate 'by hand' any function appearing in Table 4.1 on page 50, together with any constant multiples, sums, products, quotients or composites of these functions.

Note, however, that this does not absolve you from the need to be able to differentiate by hand! Just as with calculators, there are certain calculations in differentiation which are sufficiently simple (with practice) to make it worthwhile to do them by hand rather than by switching on the computer.

In more complicated cases the computer may provide the only viable option, and can then be extremely useful. As you will see, it can be used also to find precisely and to classify stationary points and to apply the Optimisation Procedure described in Subsection 2.3. Since the graph of any function can be plotted rapidly on the computer, it provides an efficient alternative to the graph-sketching techniques introduced in Subsection 2.2.

You saw a rough form of optimisation, based on a plotted graph, carried out in Chapter A3, Section 5.

Refer to Computer Book C for the work in this section.

Summary of Section 5

In this section you saw how the computer can be used to find derivatives symbolically, and you applied this facility in the context of examples which called for optimisation.

Summary of Chapter C1

In this chapter you met differentiation, which is the process that allows calculation of the rate at which one variable changes with respect to another variable. This can be visualised as the gradient of a corresponding graph. The process can be used to find the optimum value of a function over an interval, or to assist in sketching the graph of a function.

Table 4.1 on page 50 lists the derivatives of certain basic functions. Other functions may be built up from these by taking constant multiples, sums, products, quotients or composites of them. The chapter introduced rules of differentiation for all of these cases.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Calculus, gradient, tangent, smooth, rate of change (average, instantaneous), derived function, derivative, differentiation, power function, quartic function, local maximum, local minimum, stationary point, increasing/decreasing on an interval, second derived function, second derivative, optimum value, optimisation, constant function, composite function.

Symbols and notation to know and use

The various notations for first and second derivatives:

- $f'(x), f''(x)$ (function notation);
- $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d}{dx}(y), \dfrac{d}{dx}(f(x))$ (Leibniz notation);
- $\dot{s}, \ddot{s}$ (Newton's notation).

The notation for limits, as applied in the definition of the derivative:

- $f'(x) = \lim_{h \to 0}\left(\dfrac{f(x+h) - f(x)}{h}\right)$ (function notation);
- $\dfrac{dy}{dx} = \lim_{\delta x \to 0} \dfrac{\delta y}{\delta x}$ (Leibniz notation).

Mathematical skills

- Differentiate 'by hand' any of the basic functions whose derivatives are given in Table 4.1 on page 50.
- Use the appropriate rules to differentiate any constant multiple of a basic function, or any sum, product, quotient or composite of a pair of basic functions.
- Use differentiation to find instantaneous rates of change, including velocity (given a position function for an object), and evaluate the rate of change at any value of the independent variable.
- Use differentiation to find the gradient of a graph, and evaluate the gradient at any point on the graph.

- ◇ Apply the limit definition of the derived function/derivative in very simple cases.
- ◇ Differentiate twice to find second derivatives.
- ◇ Apply tests to determine whether a stationary point is a local maximum or a local minimum (or neither).
- ◇ Draw a rough sketch of the graph of a simple function, based on the location and classification of stationary points and the position of the y-intercept.
- ◇ Find the maximum or minimum value of a function on an interval.

Mathcad skills

- ◇ Apply Mathcad to differentiate an expression with respect to a specified variable, by using Mathcad's d/dx operator and symbolic evaluation.
- ◇ Use the copy and paste facilities in Mathcad.

Ideas to be aware of

- ◇ The difference between average and instantaneous rates of change.
- ◇ That the graph of a smooth function has a gradient at each point on it, as given by the gradient of the tangent at the point.
- ◇ That the derived function (derivative) of any smooth function is found by differentiation, and is equivalent both to the gradient of the graph of the function and to the instantaneous rate of change of the dependent variable with respect to the independent variable.
- ◇ Some possible uses for differentiation.
- ◇ The concept of an optimum value, and its relationship with the derivative of a function.

Solutions to Activities

Solution 1.1

In each case we apply result (1.1), which states that if $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$.

(a) For $f(x) = x^2 + x + 7$, we have $a = 1$, $b = 1$ and $c = 7$, so $f'(x) = 2x + 1$.

(b) For $f(x) = 5x^2 - x$, we have $a = 5$, $b = -1$ and $c = 0$, so $f'(x) = 10x - 1$.

(c) For $f(x) = 5 + 3x - \frac{1}{2}x^2$, we have $a = -\frac{1}{2}$, $b = 3$ and $c = 5$, so $f'(x) = -x + 3$.

Solution 1.2

(a) Result (1.1) gives $f'(x) = 2x$, so the gradient at the point $(\frac{1}{2}, \frac{1}{4})$, where $x = \frac{1}{2}$, is $f'(\frac{1}{2}) = 2 \times \frac{1}{2} = 1$.

(b) Result (1.1) gives $f'(x) = -2x$, so the gradient at the point $(3, -8)$ is $f'(3) = -2 \times 3 = -6$.

(c) Result (1.1) gives $f'(x) = 6x + 32$, so the gradient at the point $(1, -207)$ is $f'(1) = 6 \times 1 + 32 = 38$.

Solution 1.3

(a) From Example 1.1, we have

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3,$$

so we deduce that

$$\begin{aligned}(x+h)^4 &= (x+h)(x+h)^3 \\ &= (x+h)(x^3 + 3x^2h + 3xh^2 + h^3) \\ &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4.\end{aligned}$$

(b) By equation (1.3), we need to consider the quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^4 - x^4}{h},$$

where h is a non-zero number. Using the expansion for $(x+h)^4$ from part (a), we have

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= 4x^3 + 6x^2h + 4xh^2 + h^3.\end{aligned}$$

For small values of h, the expression $4x^3 + 6x^2h + 4xh^2 + h^3$ is close to $4x^3$ because each of $6x^2h$, $4xh^2$ and h^3 is small. Thus in the limit as $h \to 0$, we obtain $f'(x) = 4x^3$.

(c) The point $(\frac{1}{2}, \frac{1}{16})$ on the graph corresponds to $x = \frac{1}{2}$, for which the gradient is $f'(\frac{1}{2}) = 4(\frac{1}{2})^3 = 4(\frac{1}{8}) = \frac{1}{2}$.

Solution 1.4

In each case, we apply result (1.4), which states that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

(a) Here $f(x) = x^6$, so $n = 6$. The derivative is

$$f'(x) = 6x^{6-1} = 6x^5.$$

(b) Here $f(x) = 1/x^3 = x^{-3}$, so $n = -3$. The derivative is

$$f'(x) = -3x^{-3-1} = -3x^{-4},$$

which is often written as $f'(x) = -3/x^4$.

(c) Here $f(x) = \sqrt{x} = x^{1/2}$, so $n = \frac{1}{2}$. The derivative is

$$f'(x) = \tfrac{1}{2}x^{1/2-1} = \tfrac{1}{2}x^{-1/2},$$

which is often written as $f'(x) = 1/(2\sqrt{x})$.

Solution 1.5

(a) The position function is $s = f(t) = 5t^2$. This is a quadratic function of t, so we apply result (1.1) to obtain the velocity function

$$v = f'(t) = 10t.$$

(b) When $t = 3$, we have $v = f'(3) = 10 \times 3 = 30$, so the velocity is then $30\ \mathrm{m\,s^{-1}}$.

(c) When $v = 15$, we have $15 = 10t$, which has solution $t = 1.5$. Hence the stone has velocity $15\ \mathrm{m\,s^{-1}}$ after 1.5 seconds.

Solution 2.1

(a) From result (1.4), the derivative of x^8 is $8x^7$ and the derivative of x^5 is $5x^4$. From result (2.3), the derivative of the polynomial function $f(x) = 6x^8 - 3x^5$ is

$$\begin{aligned}f'(x) &= 6(\text{derivative of } x^8) - 3(\text{derivative of } x^5) \\ &= 6(8x^7) - 3(5x^4) \\ &= 48x^7 - 15x^4.\end{aligned}$$

(b) From results (1.4) and (2.3), the derivative of the polynomial function $f(x) = -x^3 + 5x^2 - 7x + 15$ is

$$\begin{aligned}f'(x) &= -(3x^2) + 5(2x) - 7(1) \\ &= -3x^2 + 10x - 7.\end{aligned}$$

The point $(1, 12)$ on the graph of $y = f(x)$ corresponds to $x = 1$, for which the gradient of the graph is $f'(1) = -3(1)^2 + 10(1) - 7 = 0$.

(This outcome indicates that the graph of the given function $f(x)$ has a horizontal tangent at the point $(1, 12)$.)

Solution 2.2

(a) From results (1.4) and (2.3), the derivative of the polynomial function $f(x) = x^3 - 6x^2 - 15x + 54$ is

$$\begin{aligned} f'(x) &= (3x^2) - 6(2x) - 15(1) \\ &= 3x^2 - 12x - 15 \\ &= 3(x^2 - 4x - 5) \end{aligned}$$

(b) The equation $f'(x) = 0$ is equivalent to

$$-x^2 - 4x - 5 = 0; \text{ that is, } (x+1)(x-5) = 0.$$

This equation has solutions $x = -1$ and $x = 5$. Thus the stationary points of f are at $x = -1$ and $x = 5$.

Solution 2.3

(a) For $x < -1$, we have

$$x + 1 < 0 \quad \text{and} \quad x - 5 < -6 < 0,$$

so $f'(x) = 3(x+1)(x-5) > 0$ (since $f'(x)$ is the product of 3 and of two negative numbers).

For $-1 < x < 5$, we have

$$x + 1 > 0 \quad \text{and} \quad x - 5 < 0,$$

so $f'(x) < 0$ (since $f'(x)$ is the product of 3, one other positive number and one negative number).

For $x > 5$, we have

$$x + 1 > 6 > 0 \quad \text{and} \quad x - 5 > 0,$$

so $f'(x) = 3(x+1)(x-5) > 0$ (since $f'(x)$ is the product of 3 and of two other positive numbers).

(b) By the Increasing/Decreasing Criterion, f is increasing on $(-\infty, -1)$ and on $(5, \infty)$, but decreasing on $(-1, 5)$.

Solution 2.4

For the stationary point at $x = -2$, we have to choose test points x_L to the left of -2 and x_R to the right of -2 such that there are no stationary points between x_L and -2 nor between -2 and x_R. Bearing in mind Remark 2, page 27, and the fact that the only other stationary point is at 1, we can take (say) $x_L = -3$ and $x_R = 0$. (Another choice is $x_L = -3$ and $x_R = -1$, but choosing $x_R = 0$ simplifies the calculation of $f'(x_R)$.)

For the stationary point at $x = 1$, we have to choose test points x_L to the left of 1 and x_R to the right of 1 such that there are no stationary points between x_L and 1 nor between 1 and x_R. Bearing in mind Remark 2, page 27, and the fact that the only other stationary point is at -2, we can take (say) $x_L = 0$ and $x_R = 2$.

Solution 2.5

Step 1: The derivative of the function $f(x) = 3x^4 - 2x^3 - 9x^2 + 7$ is

$$\begin{aligned} f'(x) &= 12x^3 - 6x^2 - 18x \\ &= 6x(2x^2 - x - 3) \\ &= 6x(2x - 3)(x + 1). \end{aligned}$$

Solving the equation $f'(x) = 0$, we find that the stationary points of f are at $x = -1$, $x = 0$ and $x = \frac{3}{2}$.

Step 2: To classify the stationary point at $x = -1$ we choose test points $x_L = -2$ and $x_R = -\frac{1}{2}$, say. Then we have

$$\begin{aligned} f'(x_L) &= f'(-2) = 6(-2)(-7)(-1) = -84 < 0, \\ f'(x_R) &= f'(-\tfrac{1}{2}) = 6(-\tfrac{1}{2})(-4)(\tfrac{1}{2}) = 6 > 0, \end{aligned}$$

so f has a local minimum at $x = -1$, by the First Derivative Test.

To classify the stationary point at $x = 0$, we choose test points $x_L = -\frac{1}{2}$ and $x_R = 1$, say. Then we have

$$\begin{aligned} f'(x_L) &= f'(-\tfrac{1}{2}) = 6 > 0, \\ f'(x_R) &= f'(1) = 6(1)(-1)(2) = -12 < 0, \end{aligned}$$

so f has a local maximum at $x = 0$, by the First Derivative Test. (Note that the value of $f'\left(-\frac{1}{2}\right)$ was found when classifying the stationary point at $x = -1$.)

To classify the stationary point at $x = \frac{3}{2}$, we choose test points $x_L = 1$ and $x_R = 2$, say. Then we have

$$\begin{aligned} f'(x_L) &= f'(1) = -12 < 0, \\ f'(x_R) &= f'(2) = 6(2)(1)(3) = 36 > 0, \end{aligned}$$

so f has a local minimum at $x = \frac{3}{2}$, by the First Derivative Test.

Solution 2.6

The second derivative of the function f is the (first) derivative of the function $f'(x) = -3x^2 + 10x - 7$. From results (1.4) and (2.3), this is

$$\begin{aligned} f''(x) &= -3(2x) + 10(1) \\ &= -6x + 10. \end{aligned}$$

Solution 2.7

The second derivative of f is obtained by differentiating $f'(x) = 12x^3 - 6x^2 - 18x$, to obtain

$$f''(x) = 36x^2 - 12x - 18 = 6(6x^2 - 2x - 3).$$

For the stationary point at $x = -1$, we have

$$f''(-1) = 6(6(-1)^2 - 2(-1) - 3) = 30 > 0,$$

so f has a local minimum at $x = -1$, by the Second Derivative Test.

For the stationary point at $x = 0$, we have

$$f''(0) = 6(-3) = -18 < 0,$$

so f has a local maximum at $x = 0$, by the Second Derivative Test.

For the stationary point at $x = \frac{3}{2}$, we have

$$\begin{aligned} f''(\tfrac{3}{2}) &= 6(6(\tfrac{3}{2})^2 - 2(\tfrac{3}{2}) - 3) \\ &= 6(\tfrac{27}{2} - 3 - 3) = 45 > 0, \end{aligned}$$

so f has a local minimum at $x = \frac{3}{2}$, by the Second Derivative Test.

(These classification results agree with those that you found using the First Derivative Test in Activity 2.5.)

Solution 2.8

The function is $f(x) = 3x^4 - 2x^3 - 9x^2 + 7$, and the y-coordinates of stationary points on the graph of $y = f(x)$ are

$$f(-1) = 3 + 2 - 9 + 7 = 3, \quad f(0) = 7 \quad \text{and}$$
$$f(\tfrac{3}{2}) = \tfrac{243}{16} - \tfrac{27}{4} - \tfrac{81}{4} + 7 = -\tfrac{77}{16} = -4\tfrac{13}{16}.$$

Hence the graph passes through the points $(-1, 3)$, a local minimum, $(0, 7)$, a local maximum (on the y-axis), and $(\frac{3}{2}, -4\frac{13}{16})$, a local minimum. A sketch of the graph is as follows.

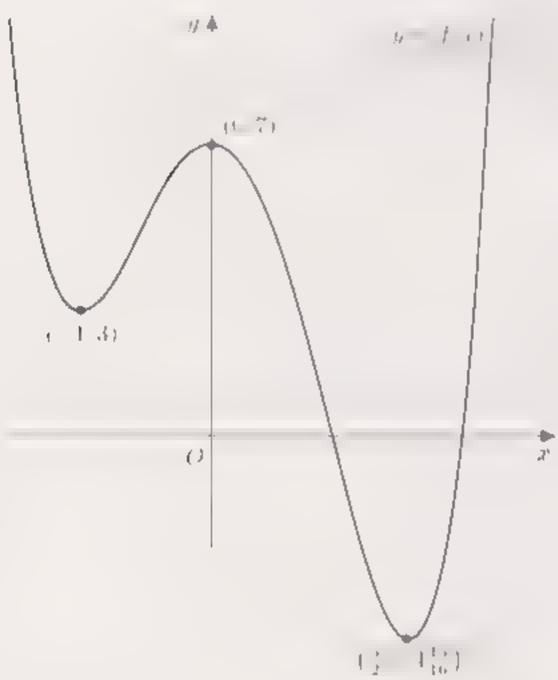


Figure S.1

Solution 2.9

In each case, we follow the steps of the Optimisation Procedure.

(a) *Step 1:* The derivative of the function $f(x) = \frac{1}{3}x^3 - x^2 - 8x + 1$ is

$$f'(x) = x^2 - 2x - 8 = (x - 4)(x + 2),$$

so the stationary points are at $x = 4$ and $x = -2$.

Step 2: At the endpoints of the interval $I = [-3, 5]$, the function values are

$$f(-3) = 7 \quad \text{and} \quad f(5) = -\tfrac{67}{3} = -22\tfrac{1}{3},$$

while at the stationary points (which are both inside I) the function values are

$$f(-2) = \tfrac{31}{3} = 10\tfrac{1}{3} \quad \text{and} \quad f(4) = -\tfrac{77}{3} = -25\tfrac{2}{3}.$$

Step 3: The greatest value of $f(x)$ on $[-3, 5]$ is $10\frac{1}{3}$ (at $x = -2$) and the least value of $f(x)$ on $[-3, 5]$ is $-25\frac{2}{3}$ (at $x = 4$).

(In this case, both of the optimum values occur at stationary points within the interval.)

(b) *Step 1:* The derivative of the function $f(x) = 3x^2 - 2x + 5$ is

$$f'(x) = 6x - 2 = 2(3x - 1),$$

so the only stationary point is at $x = \frac{1}{3}$.

Step 2: At the endpoints of the interval $I = [-1, 4]$, the function values are

$$f(-1) = 10 \quad \text{and} \quad f(4) = 45,$$

while for the stationary point (which is inside I) the function value is

$$f(\tfrac{1}{3}) = \tfrac{14}{3} = 4\tfrac{2}{3}.$$

Step 3: The greatest value of $f(x)$ on $[-1, 4]$ is 45 (at $x = 4$) and the least value of $f(x)$ on $[-1, 4]$ is $4\frac{2}{3}$ (at $x = \frac{1}{3}$).

Solution 3.1

Evaluating the ratio $(\sin h)/h$ for each of $h = 0.1, 0.01, 0.001$ in turn gives the respective answers (to 6 d.p.) 0.998 334, 0.999 983 and 1.000 000. The limiting value as $h \to 0$ appears to be 1.

Solution 3.2

Evaluating the ratio $(e^h - 1)/h$ for each of $h = 0.1, 0.001, 0.00001$ in turn gives the respective answers (to 6 d.p.) 1.051 709, 1.000 500 and 1.000 005. The limiting value as $h \to 0$ appears to be 1.

Solution 3.3

(a) The derivative of $f(x) = x^4 - 2\sin x$ is

$$f'(x) = 4x^3 - 2\cos x.$$

(b) The derivative of $g(x) = 3e^x - 4\ln x$ is

$$g'(x) = 3e^x - \frac{4}{x}.$$

(c) Since $1/x^2 = x^{-2}$ has derivative $-2x^{-3} = -2/x^3$, the derivative of $p(x) = 6/x^2 + 3\cos x$ is

$$p'(x) = -\frac{12}{x^3} - 3\sin x.$$

Solution 3.4

(a) The first and second derivatives of $V = \frac{4}{3}\pi r^3$ with respect to r are

$$\frac{dV}{dr} = 4\pi r^2 \quad \text{and} \quad \frac{d^2V}{dr^2} = 8\pi r.$$

(b) The first and second derivatives of $f(x) = 45\ln x$ are

$$f'(x) = \frac{45}{x} \quad \text{and} \quad f''(x) = -\frac{45}{x^2},$$

so we have

$$f'(5) = 9 \quad \text{and} \quad f''(3) = -5.$$

(c) The derivative of $s = 10 - 5t^2$ with respect to t is

$$\frac{ds}{dt} = -10t, \quad \text{so} \quad \left.\frac{ds}{dt}\right|_{t=2} = -20.$$

(d) The first and second derivatives of $u = 3\cos t$ with respect to t are

$$\dot{u} = -3\sin t \quad \text{and} \quad \ddot{u} = -3\cos t.$$

Solution 4.1

(a) In this case $k(x) = f(x)g(x)$, where $f(x) = x$ and $g(x) = e^x$. Since $f'(x) = 1$ and $g'(x) = e^x$, we have

$$\begin{aligned} k'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= (1)(e^x) + (x)(e^x) \\ &= e^x(1 + x). \end{aligned}$$

(b) In this case $k(x) = f(x)g(x)$, where $f(x) = 2x^2 - 1$ and $g(x) = \cos x$. Since $f'(x) = 4x$ and $g'(x) = -\sin x$, we have

$$\begin{aligned} k'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= (4x)(\cos x) + (2x^2 - 1)(-\sin x) \\ &= 4x\cos x - (2x^2 - 1)\sin x. \end{aligned}$$

(c) In this case $k(x) = f(x)g(x)$, where $f(x) = \sqrt{x} = x^{1/2}$ and $g(x) = x + \sin x$. Since $f'(x) = \frac{1}{2}x^{-1/2} = 1/(2\sqrt{x})$ and $g'(x) = 1 + \cos x$, we have

$$\begin{aligned} k'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= \left(\frac{1}{2\sqrt{x}}\right)(x + \sin x) + (\sqrt{x})(1 + \cos x) \\ &= \frac{x + \sin x}{2\sqrt{x}} + \sqrt{x}(1 + \cos x). \end{aligned}$$

This may also be written as

$$\begin{aligned} k'(x) &= \frac{x + \sin x + 2\sqrt{x}\sqrt{x}(1 + \cos x)}{2\sqrt{x}} \\ &= \frac{3x + \sin x + 2x\cos x}{2\sqrt{x}}. \end{aligned}$$

Solution 4.2

(a) The derivative of $f(x) = (3x^2 + 2x + 1)e^x$ is

$$\begin{aligned} f'(x) &= (6x + 2)(e^x) + (3x^2 + 2x + 1)(e^x) \\ &= (3x^2 + 8x + 3)e^x. \end{aligned}$$

(b) The derivative of $g(x) = \sin x \cos x$ is

$$\begin{aligned} g'(x) &= (\cos x)(\cos x) + (\sin x)(-\sin x) \\ &= \cos^2 x - \sin^2 x. \end{aligned}$$

(c) The derivative of $f(t) = (t^2 + 3)\ln t$ is

$$\begin{aligned} f'(t) &= (2t)(\ln t) + (t^2 + 3)\left(\frac{1}{t}\right) \\ &= 2t\ln t + \frac{t^2 + 3}{t}. \end{aligned}$$

Solution 4.3

(a) In this case $k(x) = f(x)/g(x)$, where $f(x) = e^x$ and $g(x) = x$. Since $f'(x) = e^x$ and $g'(x) = 1$, we have

$$\begin{aligned} k'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{(x)(e^x) - (e^x)(1)}{(x)^2} \\ &= \frac{e^x(x - 1)}{x^2}. \end{aligned}$$

(b) In this case $k(x) = f(x)/g(x)$, where $f(x) = \cos x$ and $g(x) = x^2 - 1$. Since $f'(x) = -\sin x$ and $g'(x) = 2x$, we have

$$\begin{aligned} k'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{(x^2 - 1)(-\sin x) - (\cos x)(2x)}{(x^2 - 1)^2} \\ &= -\frac{(x^2 - 1)\sin x + 2x\cos x}{(x^2 - 1)^2}. \end{aligned}$$

(c) In this case $k(x) = f(x)/g(x)$, where $f(x) = x + 1$ and $g(x) = x - 1$. Since $f'(x) = g'(x) = 1$, we have

$$\begin{aligned} k'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} \\ &= -\frac{2}{(x-1)^2}. \end{aligned}$$

Solution 4.4

(a) The derivative of $f(x) = \sin x / \cos x$ is

$$\begin{aligned} f'(x) &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \quad (\text{because } \cos^2 x + \sin^2 x = 1). \end{aligned}$$

Since $1/\cos x = \sec x$, this shows that $f'(x) = \sec^2 x$.

(b) The derivative of $g(u) = \ln u/(u^2 + 3)$ is

$$\begin{aligned} g'(u) &= \frac{(u^2+3)(1/u) - (\ln u)(2u)}{(u^2+3)^2} \\ &= \frac{u^2 + 3 - 2u^2 \ln u}{u(u^2+3)^2}. \end{aligned}$$

(c) The derivative of $p(t) = t/(e^t - 1)$ is

$$\begin{aligned} p'(t) &= \frac{(e^t-1)(1) - (t)(e^t)}{(e^t-1)^2} \\ &= \frac{(1-t)e^t - 1}{(e^t-1)^2}. \end{aligned}$$

Solution 4.5

(a) The function $k(x) = \ln(x^2 + 1)$ is of the form $k(x) = g(f(x))$ with

$$u = f(x) = x^2 + 1 \quad \text{and} \quad g(u) = \ln u.$$

(b) The function $k(x) = \cos(x^3)$ is of the form $k(x) = g(f(x))$ with

$$u = f(x) = x^3 \quad \text{and} \quad g(u) = \cos u.$$

(c) The function $k(x) = \cos^3 x = (\cos x)^3$ is of the form $k(x) = g(f(x))$ with

$$u = f(x) = \cos x \quad \text{and} \quad g(u) = u^3.$$

(Note that this case involves the same constituent functions as in part (b), but applied in the reverse order.)

(d) The function $k(x) = e^{\sin x}$ is of the form $k(x) = g(f(x))$ with

$$u = f(x) = \sin x \quad \text{and} \quad g(u) = e^u.$$

Solution 4.6

(a) In this case $k(x) = e^{x^3} = g(f(x))$, where $u = f(x) = x^3$ and $g(u) = e^u$. Now $f'(x) = 3x^2$ and $g'(u) = e^u$, so we have

$$\begin{aligned} k'(x) &= g'(f(x))f'(x) \\ &= g'(u)f'(x) \quad (\text{where } u = f(x) = x^3) \\ &= (e^u)(3x^2) \\ &= 3x^2 e^{x^3} \quad (\text{since } u = x^3). \end{aligned}$$

(b) In this case $k(x) = \sin(2x) = g(f(x))$, where $u = f(x) = 2x$ and $g(u) = \sin u$. Now $f'(x) = 2$ and $g'(u) = \cos u$, so we have

$$\begin{aligned} k'(x) &= g'(f(x))f'(x) \\ &= g'(u)f'(x) \quad (\text{where } u = f(x) = 2x) \\ &= (\cos u)(2) \\ &= 2\cos(2x) \quad (\text{since } u = 2x). \end{aligned}$$

(c) In this case $k(x) = \cos(e^x) = g(f(x))$, where $u = f(x) = e^x$ and $g(u) = \cos u$. Now $f'(x) = e^x$ and $g'(u) = -\sin u$, so we have

$$\begin{aligned} k'(x) &= g'(f(x))f'(x) \\ &= g'(u)f'(x) \quad (\text{where } u = f(x) = e^x) \\ &= (-\sin u)(e^x) \\ &= -e^x \sin(e^x) \quad (\text{since } u = e^x). \end{aligned}$$

Solution 4.7

In each case, a concise solution (obtained by 'informal' use of the Composite Rule) is followed by an expanded version. Note in the expanded versions for parts (a) and (b) that f and g cannot be used to denote the 'inner' and 'outer' functions, since one of them appears as the name of the given composite function. Hence we label the composite as $q(p(x))$ instead.

(a) The derivative of $f(x) = \cos(4x)$ is

$$\begin{aligned} f'(x) &= (-\sin(4x))(4) \\ &= -4\sin(4x). \end{aligned}$$

Expanded version: In this case $f(x) = \cos(4x) = q(p(x))$, where $u = p(x) = 4x$ and $q(u) = \cos u$. Now $p'(x) = 4$ and $q'(u) = -\sin u$, so we have

$$\begin{aligned} f'(x) &= q'(p(x))p'(x) \\ &= q'(u)p'(x) \quad (\text{where } u = p(x) = 4x) \\ &= (-\sin u)(4) \\ &= -4\sin(4x) \quad (\text{since } u = 4x). \end{aligned}$$

(b) The derivative of $g(x) = \ln(x^2 + x + 1)$ is

$$
\begin{aligned}
g'(x) &= \left(\frac{1}{x^2 + x + 1}\right)(2x + 1) \\
&= \frac{2x + 1}{x^2 + x + 1}
\end{aligned}
$$

Expanded version: In this case $g(x) = \ln(x^2 + x + 1) = q(p(x))$, where $u = p(x) = x^2 + x + 1$ and $q(u) = \ln u$. Now $p'(x) = 2x + 1$ and $q'(u) = 1/u$, so we have

$$
\begin{aligned}
g'(x) &= q'(p(x))p'(x) \\
&= q'(u)p'(x) \\
&\quad \text{(where } u = p(x) = x^2 + x + 1\text{)} \\
&= \left(\frac{1}{u}\right)(2x + 1) \\
&= \frac{2x + 1}{x^2 + x + 1} \quad \text{(since } u = x^2 + x + 1\text{).}
\end{aligned}
$$

(c) The derivative of $v(t) = \sin(\cos t)$ is

$$
\begin{aligned}
v'(t) &= (\cos(\cos t))(-\sin t) \\
&= -\sin t \cos(\cos t).
\end{aligned}
$$

Expanded version: In this case $v(t) = \sin(\cos t) = q(p(t))$, where $u = p(t) = \cos t$ and $q(u) = \sin u$. Now $p'(t) = -\sin t$ and $q'(u) = \cos u$, so we have

$$
\begin{aligned}
v'(t) &= q'(p(t))p'(t) \\
&= q'(u)p'(t) \quad \text{(where } u = p(t) = \cos t\text{)} \\
&= (\cos u)(-\sin t) \\
&= -\sin t \cos(\cos t) \quad \text{(since } u = \cos t\text{).}
\end{aligned}
$$

Solutions to Exercises

Solution 1.1

Result (1.1) states that if $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$. For $y = f(x) = 6 + 5x - 4x^2$, we have $a = -4$, $b = 5$ and $c = 6$, so that

$$f'(x) = -8x + 5.$$

Hence the gradient at the point $(2, 0)$, where $x = 2$, is $f'(2) = -8 \times 2 + 5 = -11$.

Solution 1.2

(a) Putting the left-hand side fractions over the common denominator $(x+h)x$, we have

$$\begin{aligned}\frac{1}{x+h} - \frac{1}{x} &= \frac{x}{(x+h)x} - \frac{x+h}{(x+h)x}\\ &= \frac{x-(x+h)}{(x+h)x}\\ &= \frac{-h}{(x+h)x},\end{aligned}$$

as required.

(b) By equation (1.3), and using the result of part (a), we need to consider the quotient

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)\\ &= \frac{1}{h}\left(\frac{-h}{(x+h)x}\right)\\ &= -\frac{1}{(x+h)x}.\end{aligned}$$

For small values of h, the quotient $-1/((x+h)x)$ is close to $-1/x^2$. We conclude that

$$f'(x) = -\frac{1}{x^2}.$$

(c) Result (1.4) states that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. Here $f(x) = 1/x = x^{-1}$, so that $n = -1$. The derivative is therefore

$$f'(x) = -1x^{-1-1} = -x^{-2},$$

which is often written as $f'(x) = -1/x^2$, in agreement with the result of part (b).

Solution 1.3

(a) The position function is $s = f(t) = 3t^2 + t$. This is a quadratic function of t, so we apply result (1.1) to obtain the velocity function

$$v = f'(t) = 6t + 1.$$

(b) When $t = 10$, we have $v = f'(10) = 6 \times 10 + 1 = 61$, so the velocity is then $61\ \mathrm{m\,s^{-1}}$.

(c) When $v = 25$, we have $25 = 6t + 1$, which has solution $t = 4$. Hence the aeroplane has velocity $25\ \mathrm{m\,s^{-1}}$ after 4 seconds.

Solution 2.1

(a) From result (1.4), the derivative of x^{101} is $101x^{100}$, the derivative of x^{50} is $50x^{49}$ and the derivative of x $(= x^1)$ is 1 $(= x^0)$. From result (2.3), the derivative of the polynomial function $f(x) = 7x^{101} - 13x^{50} + x$ is

$$\begin{aligned}f'(x) &= 7(101x^{100}) - 13(50x^{49}) + 1\\ &= 707x^{100} - 650x^{49} + 1.\end{aligned}$$

(b) From results (1.4) and (2.3), the derivative of the polynomial function $f(x) = 5x^3 + 4x^2 - 9x - 8$ is

$$\begin{aligned}f'(x) &= 5(3x^2) + 4(2x) - 9(1)\\ &= 15x^2 + 8x - 9.\end{aligned}$$

The point $(-1, 0)$ on the graph of $y = f(x)$ corresponds to $x = -1$, for which the gradient is $f'(-1) = 15(-1)^2 + 8(-1) - 9 = -2$.

Solution 2.2

(a) The derivative of the function $f(x) = -x^3 - 3x^2 + 9x + 5$ is

$$f'(x) = -3x^2 - 6x + 9 = -3(x+3)(x-1).$$

Solving the equation $f'(x) = 0$, we find that the stationary points of f are at $x = -3$ and $x = 1$.

(b) (i) Step 1 of the strategy has been carried out in part (a). Step 2 is as follows.

To classify the stationary point at $x = -3$, we choose test points $x_L = -4$ and $x_R = 0$, say. Then we have

$$\begin{aligned}f'(x_L) &= f'(-4) = -3(-1)(-5) = -15 < 0,\\ f'(x_R) &= f'(0) = 9 > 0,\end{aligned}$$

so f has a local minimum at $x = -3$, by the First Derivative Test.

To classify the stationary point at $x = 1$, we choose test points $x_L = 0$ and $x_R = 2$, say. Then we have

$$\begin{aligned}f'(x_L) &= f'(0) = 9 > 0,\\ f'(x_R) &= f'(2) = -3(5)(1) = -15 < 0,\end{aligned}$$

so f has a local maximum at $x = 1$, by the First Derivative Test.

(ii) The second derivative of f is

$$f''(x) = -6x - 6 = -6(x+1).$$

For the stationary point at $x = -3$, we have

$$f''(-3) = -6(-2) = 12 > 0,$$

so f has a local minimum at $x = -3$, by the Second Derivative Test.

For the stationary point at $x = 1$, we have

$$f''(1) = -6(2) = -12 < 0,$$

so f has a local maximum at $x = 1$, by the Second Derivative Test.

(c) The y-coordinates of stationary points on the graph of $y = f(x)$ are

$$f(-3) = 27 - 27 - 27 + 5 = -22,$$
$$f(1) = -1 - 3 + 9 + 5 = 10$$

Also $f(0) = 5$. Hence the graph passes through the points $(-3, -22)$, a local minimum, and $(1, 10)$, a local maximum. It also passes through $(0, 5)$; that is, it cuts the y-axis at $y = 5$. A sketch of the graph is as follows.

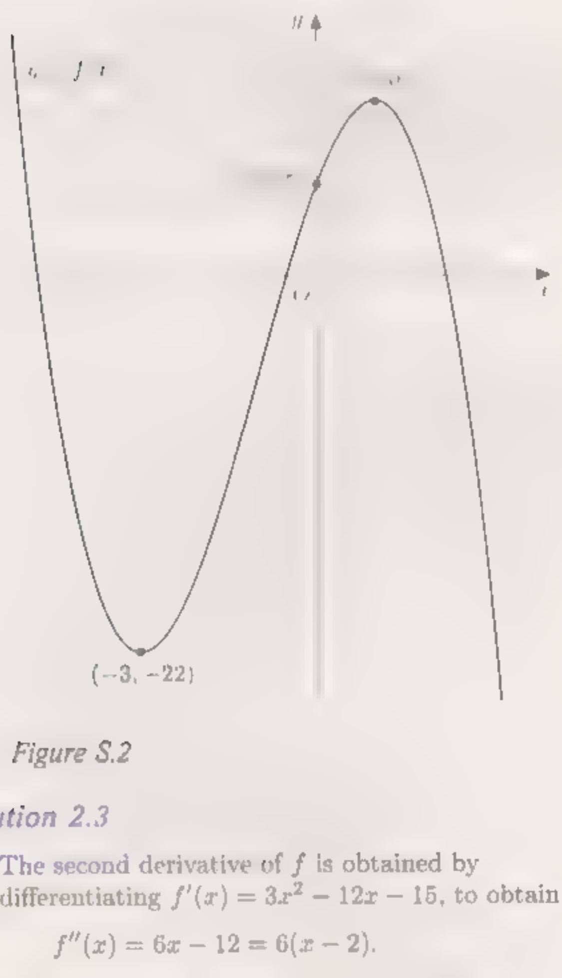


Figure S.2

Solution 2.3

(a) The second derivative of f is obtained by differentiating $f'(x) = 3x^2 - 12x - 15$, to obtain

$$f''(x) = 6x - 12 = 6(x - 2).$$

For the stationary point at $x = -1$, we have

$$f''(-1) = 6(-3) = -18 < 0,$$

so f has a local maximum at $x = -1$, by the Second Derivative Test.

For the stationary point at $x = 5$, we have

$$f''(5) = 6(3) = 18 > 0,$$

so f has a local minimum at $x = 5$, by the Second Derivative Test.

(b) In each case, we follow the steps of the Optimisation Procedure, but Step 1 has already been completed. The stationary points of the function $f(x) = x^3 - 6x^2 - 15x + 54$ are at $x = -1$ and $x = 5$.

(i) *Step 2:* At the endpoints of the interval $I = [0, 7]$, the function values are

$$f(0) = 54 \quad \text{and} \quad f(7) = -2,$$

while for the stationary point at $x = 5$ (which is inside I) the function value is

$$f(5) = -46.$$

(Note that, on this occasion, the other stationary point (at $x = -1$) is *not* inside I, and hence does not feature at this stage.)

Step 3: The greatest value of $f(x)$ on $[0, 7]$ is 54 (at $x = 0$) and the least value of $f(x)$ on $[0, 7]$ is -46 (at $x = 5$).

(ii) *Step 2:* At the endpoints of the interval $I = [-4, 7]$, the function values are

$$f(-4) = -46 \quad \text{and} \quad f(7) = -2,$$

while for the stationary points (both of which are inside I) the function values are

$$f(-1) = 62 \quad \text{and} \quad f(5) = -46.$$

Step 3: The greatest value of $f(x)$ on $[-4, 7]$ is 62 (at $x = -1$) and the least value of $f(x)$ on $[-4, 7]$ is -46 (at both $x = -4$ and $x = 5$).

Solution 3.1

(a) The derivative of $u(x) = 4\sin x - 5\cos x$ is

$$u'(x) = 4\cos x + 5\sin x.$$

(b) The derivative of $v(x) = 7\ln x - 2e^x$ is

$$v'(x) = \frac{7}{x} - 2e^x.$$

(c) Since $1/x = x^{-1}$ has derivative $-x^{-2}$ ..., and $1/x^2 = x^{-2}$ has derivative $-2x^{-}$ ..., the derivative of $w(x) = 3x^2 - 2x + 1 - 1/x + 2/x^2$ is

$$w'(x) = 6x - 2 + \frac{1}{x^2} - \frac{1}{x^3}.$$

Solution 3.2

(a) The first and second derivatives of $P = \cos u - \sin u$ with respect to u are

$$\frac{dP}{du} = -\sin u - \cos u, \qquad \frac{d^2P}{du^2} = -\cos u + \sin u.$$

(b) The first and second derivatives of $v = -e^t + 17t + 3$ with respect to t are

$$\dot{v} = -e^t + 17, \quad \ddot{v} = -e^t$$

(c) The derivative of $w = 5s^4 - 10s^2$ with respect to s is

$$\frac{dw}{ds} = 20s^3 - 20s, \quad \text{so} \quad \left.\frac{dw}{ds}\right|_{s=3} = 480.$$

Solution 4.1

(a) The derivative of $f(x) = (x^3 + x)\sin x$ is

$$\begin{aligned} f'(x) &= (3x^2 + 1)(\sin x) + (x^3 + x)(\cos x) \\ &= (3x^2 + 1)\sin x + (x^3 + x)\cos x. \end{aligned}$$

(b) The derivative of $g(x) = e^x \cos x$ is

$$\begin{aligned} g'(x) &= (e^x)(\cos x) + (e^x)(-\sin x) \\ &= e^x(\cos x - \sin x). \end{aligned}$$

(c) The derivative of $\sqrt{t} = t^{1/2}$ is $\frac{1}{2}t^{-1/2} = 1/(2\sqrt{t})$. Hence the derivative of $u(t) = \sqrt{t}\ln t$ is

$$\begin{aligned} u'(t) &= \left(\frac{1}{2\sqrt{t}}\right)(\ln t) + \left(\sqrt{t}\right)\left(\frac{1}{t}\right) \\ &= \frac{\ln t + 2}{2\sqrt{t}}. \end{aligned}$$

Solution 4.2

(a) The derivative of $g(x) = x^4/\ln x$ is

$$\begin{aligned} g'(x) &= \frac{(\ln x)(4x^3) - (x^4)(1/x)}{(\ln x)^2} \\ &= \frac{x^3(4\ln x - 1)}{(\ln x)^2}. \end{aligned}$$

(b) The derivative of $k(t) = (\sin t - t^2)/e^t$ is

$$\begin{aligned} k'(t) &= \frac{(e^t)(\cos t - 2t) - (\sin t - t^2)(e^t)}{(e^t)^2} \\ &= \frac{\cos t - \sin t + t^2 - 2t}{e^t}. \end{aligned}$$

(c) The derivative of $f(x) = \dfrac{1}{\cos x}$ is

$$\begin{aligned} f'(x) &= \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} \\ &= \frac{\sin x}{\cos^2 x}. \end{aligned}$$

(Since $1/\cos x = \sec x$ and $\sin x/\cos x = \tan x$, this shows that the derivative of $f(x) = \sec x$ is $f'(x) = \sec x \tan x$.)

Solution 4.3

(a) The derivative of $k(x) = \sin(\ln x)$ is

$$k'(x) = (\cos(\ln x))\left(\frac{1}{x}\right) = \frac{\cos(\ln x)}{x}.$$

Expanded version: In this case $k(x) = \sin(\ln x) = g(f(x))$, where $u = f(x) = \ln x$ and $g(u) = \sin u$. Now $f'(x) = 1/x$ and $g'(u) = \cos u$, so we have

$$\begin{aligned} k'(x) &= g'(f(x))f'(x) \\ &= g'(u)f'(x) \quad (\text{where } u = f(x) = \ln x) \\ &= (\cos u)\left(\frac{1}{x}\right) \\ &= \frac{\cos(\ln x)}{x} \quad (\text{since } u = \ln x). \end{aligned}$$

(b) The derivative of $k(x) = \sin^4 x = (\sin x)^4$ is

$$k'(x) = \left(4(\sin x)^3\right)(\cos x) = 4\cos x \sin^3 x.$$

Expanded version: In this case $k(x) = (\sin x)^4 = g(f(x))$, where $u = f(x) = \sin x$ and $g(u) = u^4$. Now $f'(x) = \cos x$ and $g'(u) = 4u^3$, so we have

$$\begin{aligned} k'(x) &= g'(f(x))f'(x) \\ &= g'(u)f'(x) \quad (\text{where } u = f(x) = \sin x) \\ &= (4u^3)(\cos x) \\ &= 4\cos x \sin^3 x \quad (\text{since } u = \sin x). \end{aligned}$$

(c) The derivative of $k(x) = e^{\sqrt{x}}$ is

$$k'(x) = \left(e^{\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}.$$

Expanded version: In this case $k(x) = e^{\sqrt{x}} = g(f(x))$, where $u = f(x) = \sqrt{x} = x^{1/2}$ and $g(u) = e^u$. Now $f'(x) = \frac{1}{2}x^{-1/2} = 1/\left(2\sqrt{x}\right)$ and $g'(u) = e^u$, so we have

$$\begin{aligned} k'(x) &= g'(f(x))f'(x) \\ &= g'(u)f'(x) \quad (\text{where } u = f(x) = \sqrt{x}) \\ &= e^u\left(\frac{1}{2\sqrt{x}}\right) \\ &= \frac{e^{\sqrt{x}}}{2\sqrt{x}} \quad (\text{since } u = \sqrt{x}). \end{aligned}$$

(d) The derivative of $p(t) = \ln(e^t + t)$ is

$$p'(t) = \left(\frac{1}{e^t + t}\right)(e^t + 1) = \frac{e^t + 1}{e^t + t}.$$

Expanded version: In this case
$p(t) = \ln(e^t + t) = g(f(t))$, where
$u = f(t) = e^t + t$ and $g(u) = \ln u$. Now
$f'(t) = e^t + 1$ and $g'(u) = 1/u$, so we have

$$\begin{aligned} p'(t) &= g'(f(t))f'(t) \\ &= g'(u)f'(t) \quad (\text{where } u = f(t) = e^t + t) \\ &= \left(\frac{1}{u}\right)(e^t + 1) \\ &= \frac{e^t + 1}{e^t + t} \quad (\text{since } u = e^t + t). \end{aligned}$$

(e) The derivative of $q(s) = \cos(1/s^2)$ is

$$q'(s) = \left(-\sin\left(\frac{1}{s^2}\right)\right)\left(-\frac{2}{s^3}\right) = \frac{2\sin(1/s^2)}{s^3}.$$

Expanded version: In this case
$q(s) = \cos(1/s^2) = g(f(s))$, where
$u = f(s) = 1/s^2 = s^{-2}$ and $g(u) = \cos u$. Now
$f'(s) = -2s^{-3} = -2/s^3$ and $g'(u) = -\sin u$, so
we have

$$\begin{aligned} q'(s) &= g'(f(s))f'(s) \\ &= g'(u)f'(s) \quad (\text{where } u = f(s) = 1/s^2) \\ &= (-\sin u)\left(-\frac{2}{s^3}\right) \\ &= \frac{2\sin(1/s^2)}{s^3} \quad (\text{since } u = 1/s^2). \end{aligned}$$

(f) The derivative of $r(y) = 1/\cos^2 y$ is

$$r'(y) = \left(-\frac{2}{\cos^3 y}\right)(-\sin y) = \frac{2\sin y}{\cos^3 y}.$$

Expanded version: In this case
$r(y) = 1/\cos^2 y = g(f(y))$, where
$u = f(y) = \cos y$ and $g(u) = 1/u^2 = u^{-2}$. Now
$f'(y) = -\sin y$ and $g'(u) = -2u^{-3} = -2/u^3$, so
we have

$$\begin{aligned} r'(y) &= g'(f(y))f'(y) \\ &= g'(u)f'(y) \quad (\text{where } u = f(y) = \cos y) \\ &= \left(-\frac{2}{u^3}\right)(-\sin y) \\ &= \frac{2\sin y}{\cos^3 y} \quad (\text{since } u = \cos y). \end{aligned}$$

(Note that this case involves the same constituent functions as in part (e), but applied in the reverse order.)

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